Generalised hydrogen interactions with CINCO

Jack Shergold

& Martin Bauer, Javier Pérez-Soler; 2407.12913

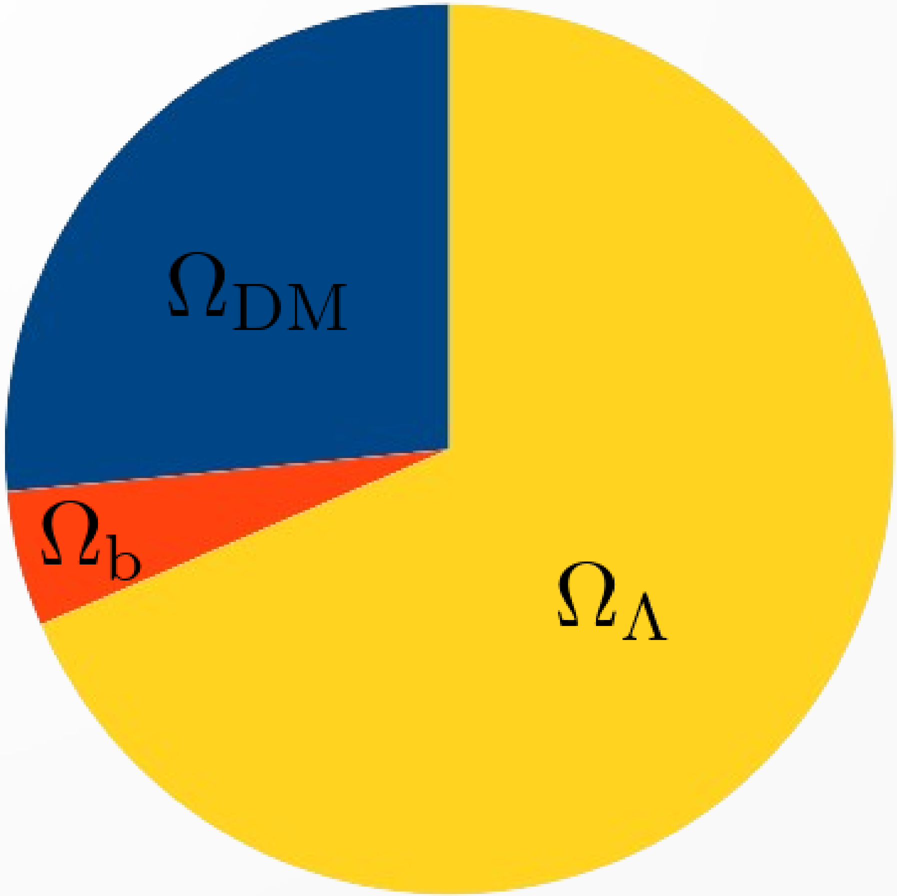
To appear in JHEP

Contents

- Generalised hydrogen interactions
  - The relativistic hydrogen atom
  - Transition rates
- CINCO
- What's next?

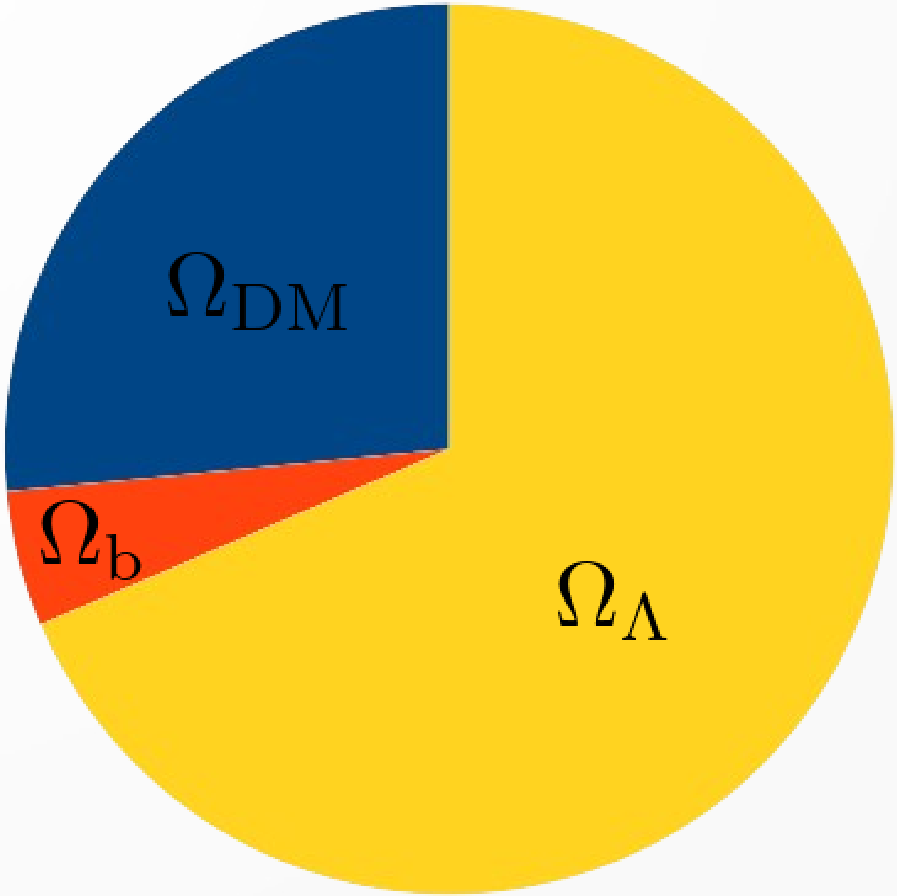
Generalised hydrogen interactions

- Gravitional evidence for dark matter on many scales



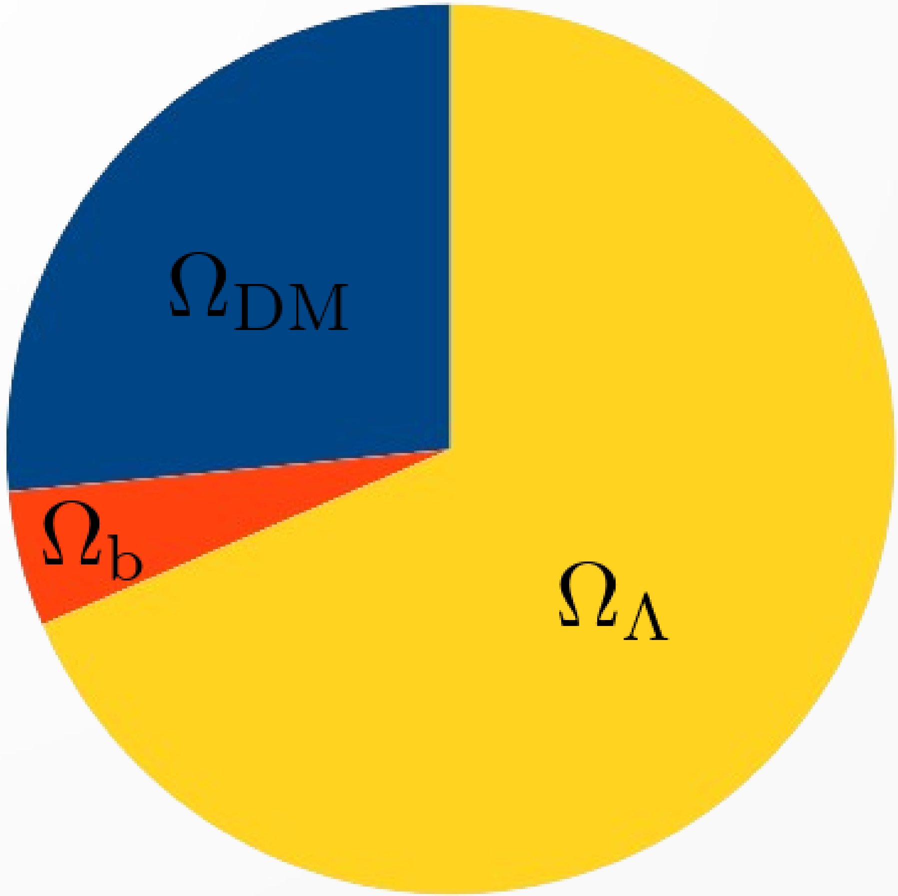
Generalised hydrogen interactions

- Gravitional evidence for dark matter on many scales
- Estimated to make up ~27% of the universe



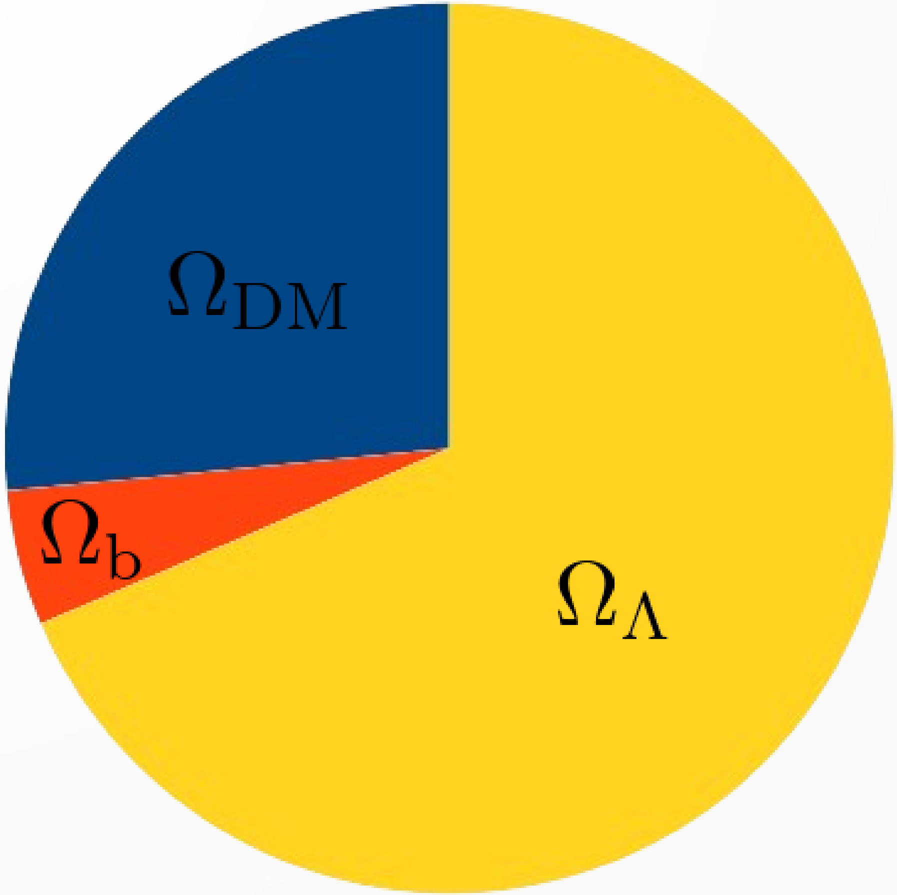
Generalised hydrogen interactions

- Gravitional evidence for dark matter on many scales
- Estimated to make up ~27% of the universe
- Microscopic nature completely unknown



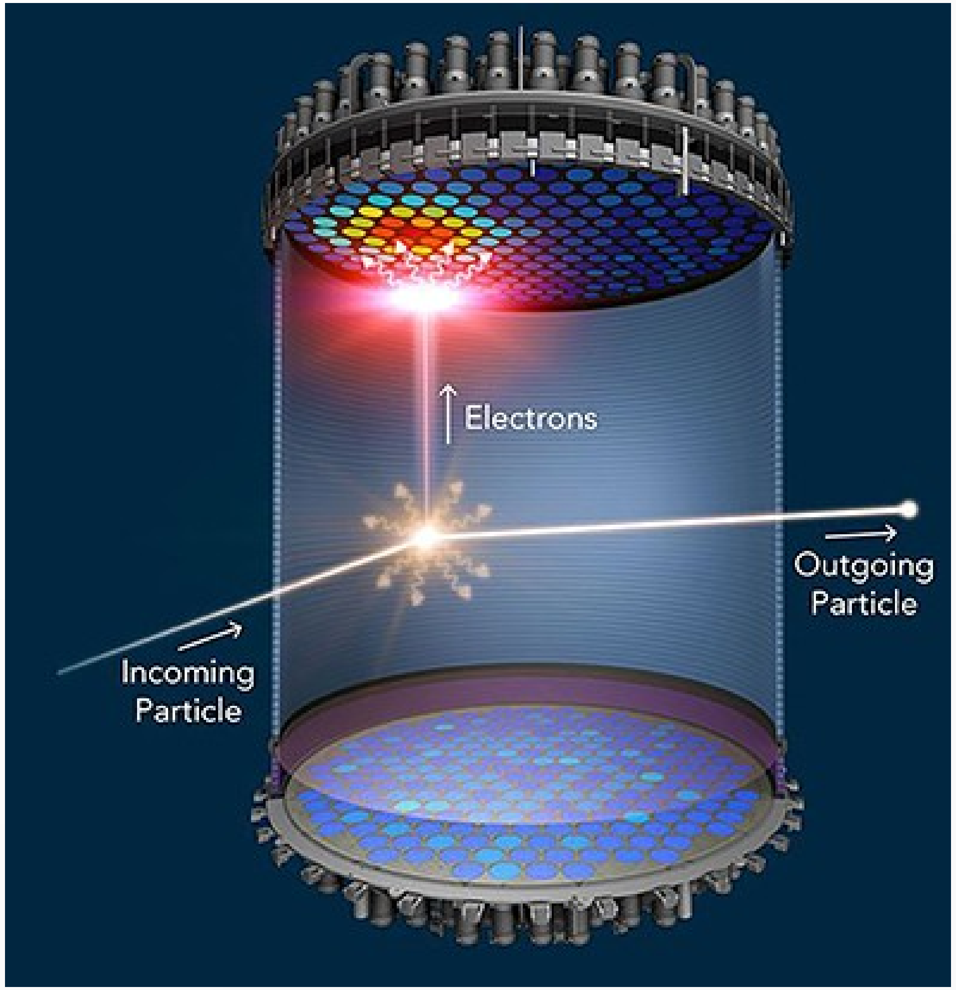
Generalised hydrogen interactions

- Gravitional evidence for dark matter on many scales
- Estimated to make up ~27% of the universe
- Microscopic nature completely unknown
- How do we search for it?



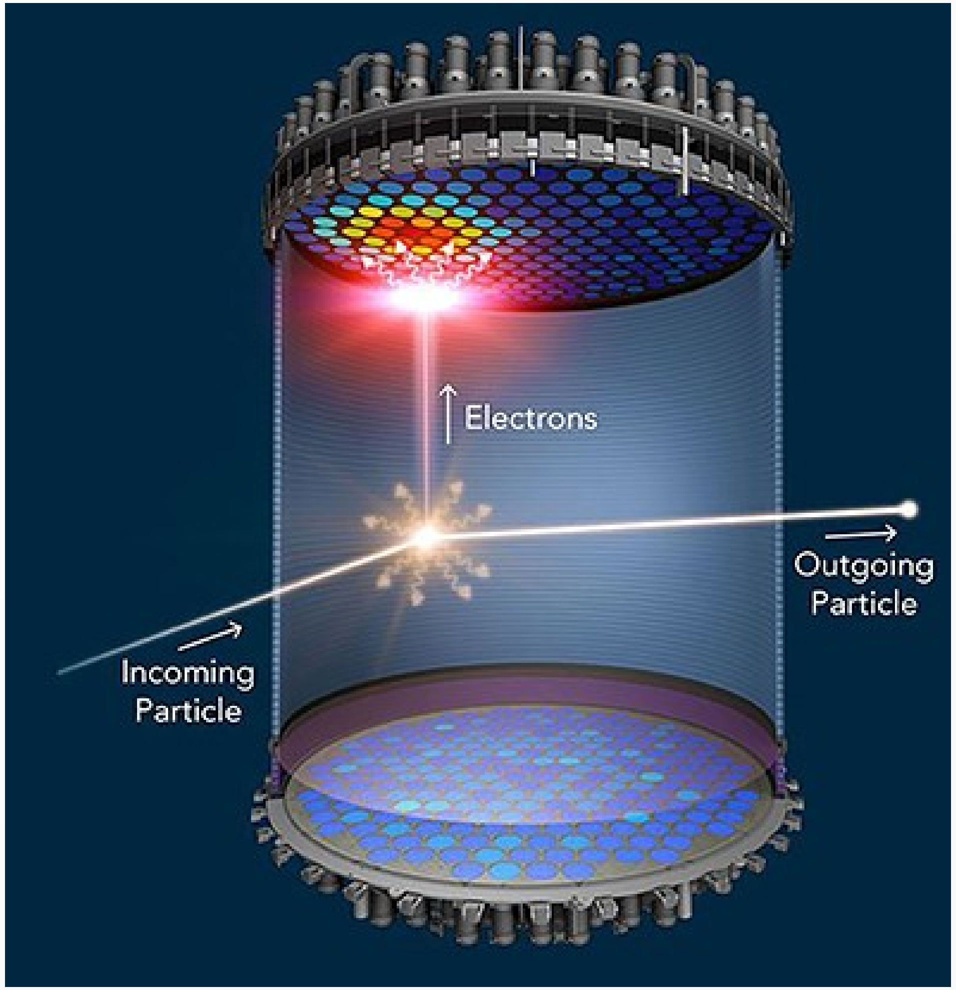
Generalised hydrogen interactions

- Cryogenic detectors



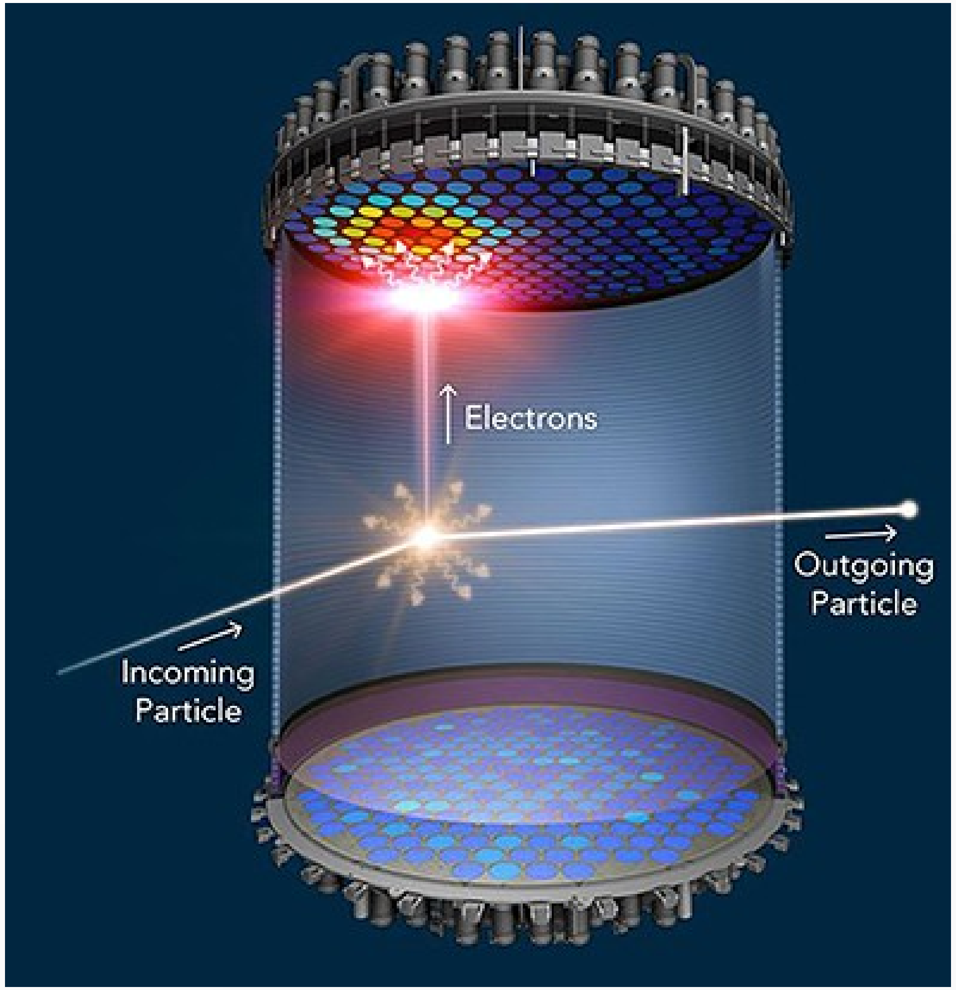
Generalised hydrogen interactions

- Cryogenic detectors
- Spin precession experiments



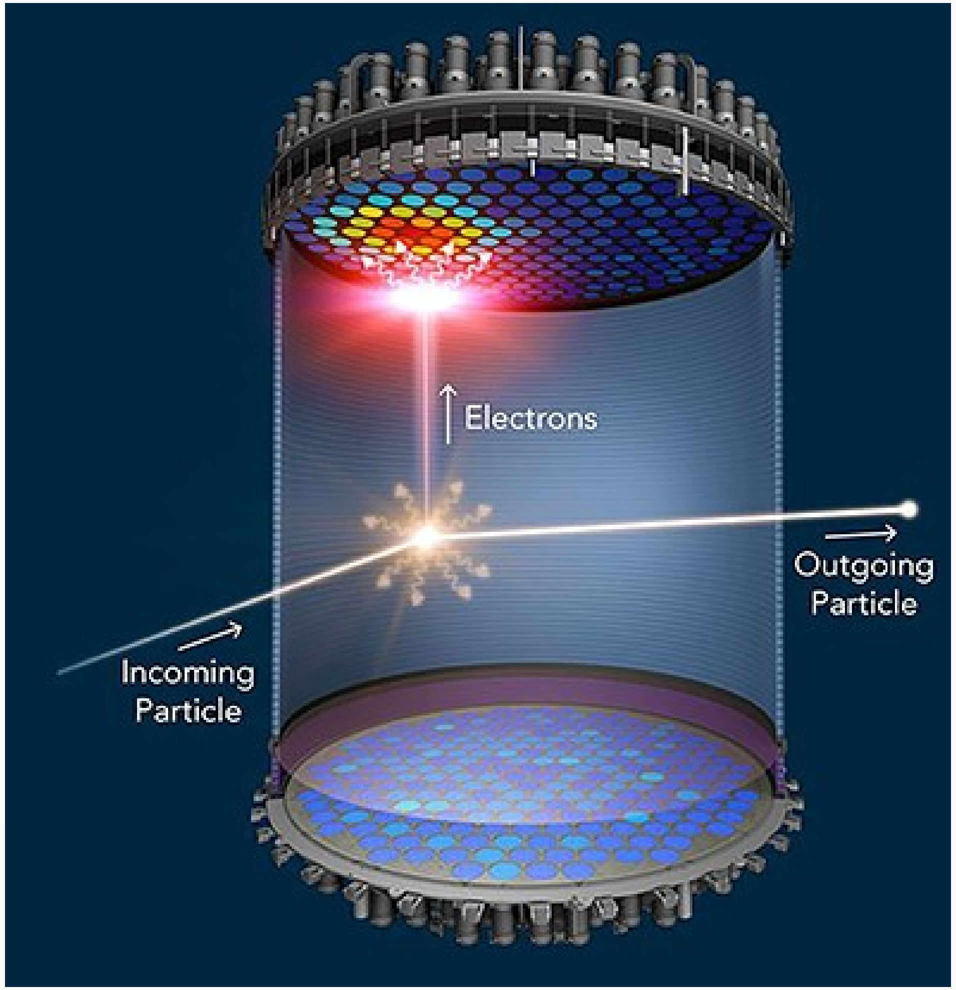
Generalised hydrogen interactions

- Cryogenic detectors
- Spin precession experiments
- Atom interometers



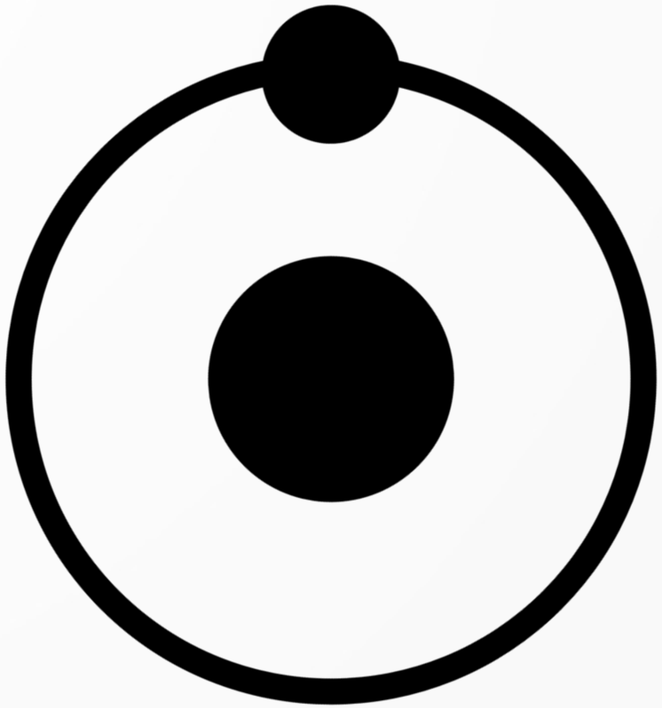
Generalised hydrogen interactions

- Cryogenic detectors
- Spin precession experiments
- Atom interometers
- …



Generalised hydrogen interactions

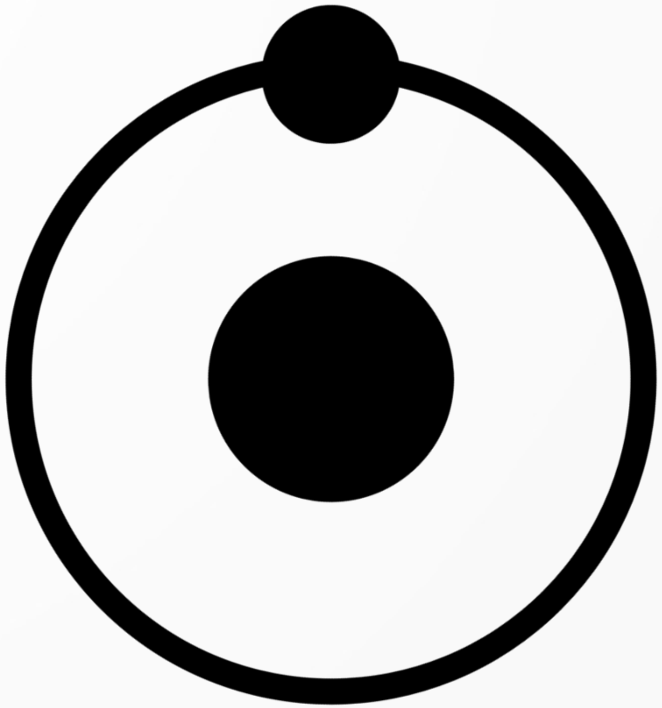
- Hydrogen is one of the best studied QM systems



Generalised hydrogen interactions

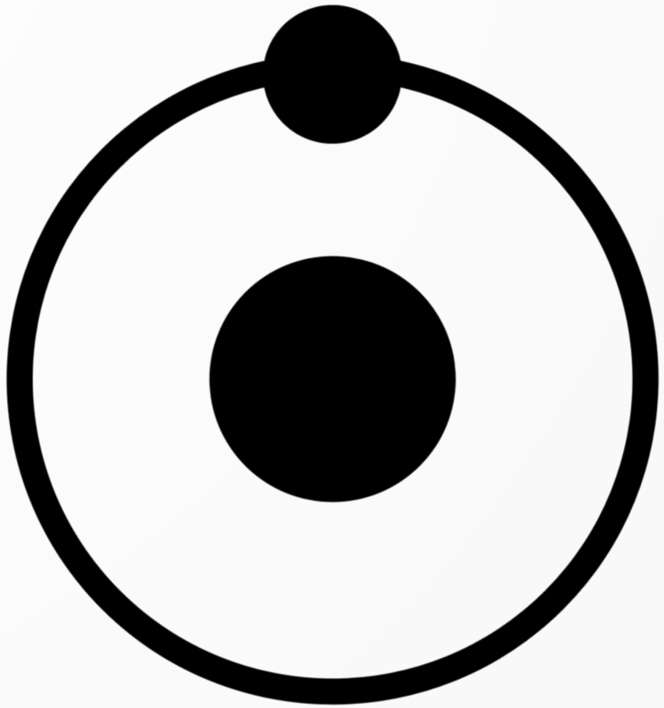
- Hydrogen is one of the best studied QM systems

- Hydrogen is a very *clean* laboratory



Generalised hydrogen interactions

- Hydrogen is one of the best studied QM systems
- Hydrogen is a very *clean* laboratory
- Sensitive to new physics!



Generalised hydrogen interactions

- Where is the dark matter?

Generalised hydrogen interactions

- Where is the dark matter?

- Long range forces:

$$U(r) = \frac{Z\alpha_{\text{EM}}}{r}$$

Generalised hydrogen interactions

- Where is the dark matter?

- Long range forces:

$$U(r) = \frac{Z\alpha_{\text{EM}}}{r} \longrightarrow U_{\text{NP}}(r) = \frac{Z\alpha_{\text{EM}}}{r} + \beta\mathcal{R}(r)$$

Generalised hydrogen interactions

- Where is the dark matter?

- Long range forces:

$$U(r) = \frac{Z\alpha_{\mathrm{EM}}}{r} \longrightarrow U_{\mathrm{NP}}(r) = \frac{Z\alpha_{\mathrm{EM}}}{r} + \beta\mathcal{R}(r)$$

- In general:

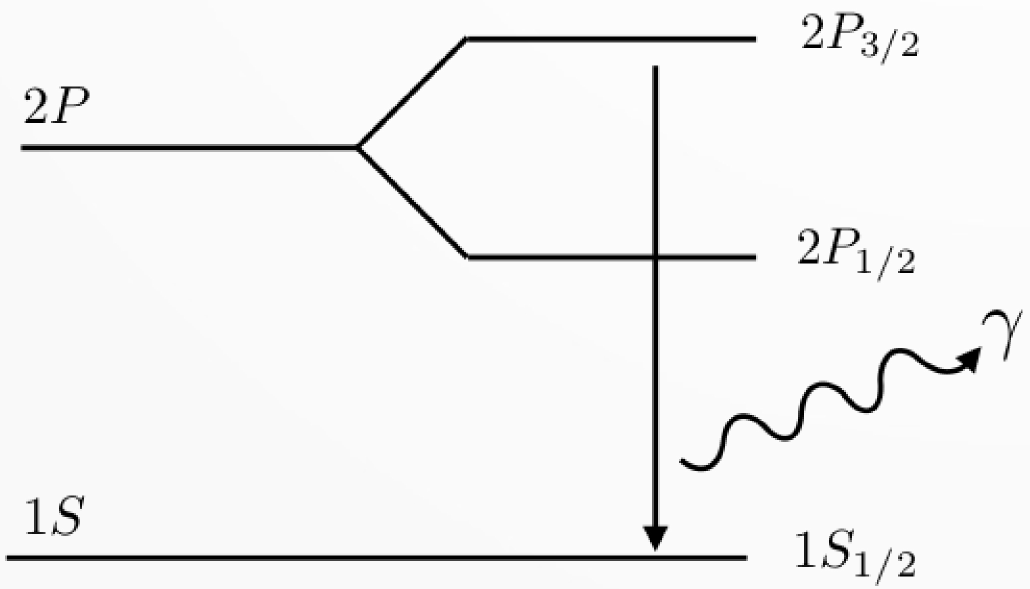
$$\mathcal{R}(r) = \frac{1}{r^n}e^{-mr}$$

Generalised hydrogen interactions

- Where is the dark matter?
- New transitions:

Generalised hydrogen interactions

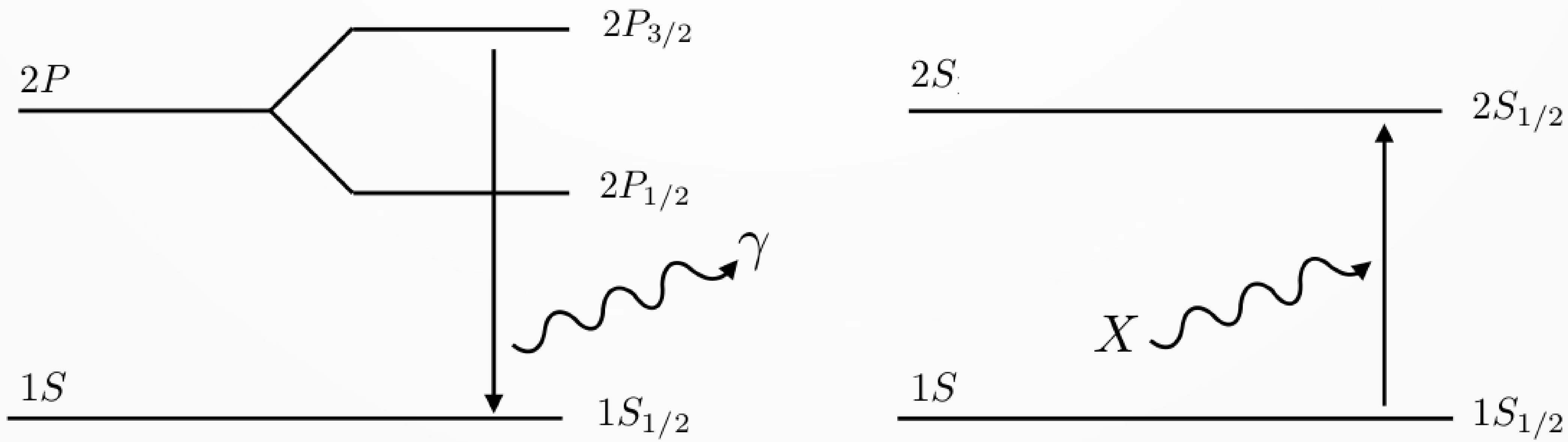
- Where is the dark matter?
- New transitions:



Generalised hydrogen interactions

- Where is the dark matter?

- New transitions:



Generalised hydrogen interactions

- Where is the dark matter?

Generalised hydrogen interactions

- Where is the dark matter?
- These transitions scale strongly with Z :

Generalised hydrogen interactions

- Where is the dark matter?

- These transitions scale strongly with Z:

$$[\Gamma] = 1$$

Generalised hydrogen interactions

- Where is the dark matter?

- These transitions scale strongly with Z:

$$[\Gamma] = 1 \qquad\qquad \Gamma \sim \sigma$$

Generalised hydrogen interactions

- Where is the dark matter?

- These transitions scale strongly with Z:

$$[\Gamma] = 1 \qquad \Gamma \sim \sigma \qquad \sigma \sim G_F^2$$

Generalised hydrogen interactions

- Where is the dark matter?

- These transitions scale strongly with Z:

$$[\Gamma] = 1 \qquad \Gamma \sim \sigma \qquad \sigma \sim G_F^2 \qquad [G_F] = -2$$

Generalised hydrogen interactions

- Where is the dark matter?

- These transitions scale strongly with Z:

$$[\Gamma] = 1 \qquad \Gamma \sim \sigma \qquad \sigma \sim G_F^2 \qquad [G_F] = -2$$

$$\Gamma \sim \Delta E_{fi}^5$$

Generalised hydrogen interactions

- Where is the dark matter?

- These transitions scale strongly with Z:

$$[\Gamma] = 1 \qquad \Gamma \sim \sigma \qquad \sigma \sim G_F^2 \qquad [G_F] = -2$$

$$\Gamma \sim \Delta E_{fi}^5 \qquad \Delta E_{fi} \sim Z^2$$

Generalised hydrogen interactions

- Where is the dark matter?

- These transitions scale strongly with Z:

$$\Gamma \sim Z^{10}$$

Generalised hydrogen interactions

- Where is the dark matter?
- These transitions scale strongly with Z:

$$\Gamma \sim Z^{10}$$

- Sensitivity to metallicity in astrophysical processes

Generalised hydrogen interactions

- Fermi's golden rule:

$$d\Gamma = 2\pi |\mathcal{M}_{fi}|^2 \delta \left(\sum_i E_i - \sum_f E_f \right) d\rho$$

Generalised hydrogen interactions

- Fermi's golden rule:

$$d\Gamma = 2\pi|\mathcal{M}_{fi}|^2\delta\left(\sum_i E_i - \sum_f E_f\right) d\rho$$

- In general:

$$\mathcal{M}_{fi} = \langle f| \int d^3x\,\mathcal{H}_{\text{int}}|i\rangle$$

Generalised hydrogen interactions

- Fermi's golden rule:

$$d\Gamma = 2\pi |\mathcal{M}_{fi}|^2 \delta \left(\sum_i E_i - \sum_f E_f \right) d\rho$$

- In general:

$$\mathcal{M}_{fi} = \langle f | \int d^3x \, \mathcal{H}_{\text{int}} | i \rangle$$

$$\mathcal{H}_{\text{int}} = \mathcal{H}_{\text{int}}(\psi_e, A_\mu, \phi, \ldots)$$

Contents

- Generalised hydrogen interactions
  - The relativistic hydrogen atom ⇦
  - Transition rates
- CINCO
- What’s next?

The relativistic hydrogen atom

- Why the relativistic hydrogen atom?

The relativistic hydrogen atom

- Why the relativistic hydrogen atom?

- Because it's easier…

The relativistic hydrogen atom

- Why the relativistic hydrogen atom?

- Because it’s easier…lets us use the language of QFT:

The relativistic hydrogen atom

- Why the relativistic hydrogen atom?

- Because it's easier...lets us use the language of QFT:

$$A_\mu \bar{\psi}_e \gamma^\mu \psi_e$$

The relativistic hydrogen atom

- Why the relativistic hydrogen atom?

- Because it's easier...lets us use the language of QFT:

$$A_\mu \bar{\psi}_e \gamma^\mu \psi_e \xrightarrow{\text{FW}} \vec{p}_e \cdot \vec{A}$$

The relativistic hydrogen atom

- Why the relativistic hydrogen atom?

- Because it's easier...lets us use the language of QFT:

$$A_\mu \bar{\psi}_e \gamma^\mu \psi_e \xrightarrow{\text{FW}} \vec{p}_e \cdot \vec{A}$$

The relativistic hydrogen atom

- Why the relativistic hydrogen atom?

- Because it's easier...lets us use the language of QFT:

$$A_\mu \bar{\psi}_e \gamma^\mu \psi_e \xrightarrow{\text{FW}} \vec{p}_e \cdot \vec{A}$$

- More precise results at large Z

The relativistic hydrogen atom

- Why (not) the relativistic hydrogen atom?

The relativistic hydrogen atom

- Why (not) the relativistic hydrogen atom?

- Four components → much more computation

The relativistic hydrogen atom

- Why (not) the relativistic hydrogen atom?
- Four components → much more computation
- Classical labelling (E1, M1, ...) obscured

The relativistic hydrogen atom

- The Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - \gamma^0 \frac{Z\alpha_{\text{EM}}}{r} - \mu\right)\mathcal{U} = 0$$

The relativistic hydrogen atom

- The Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - \gamma^0 \frac{Z\alpha_{\mathrm{EM}}}{r} - \mu \right) \mathcal{U} = 0$$

- Spherical symmetry:

$$\mathcal{U} \propto Y_{j,m}(\theta, \phi)$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\Omega_{j,l,m}(\theta,\phi) \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta,\phi) \end{pmatrix}$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\boxed{\Omega_{j,l,m}(\theta,\phi)} \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta,\phi) \end{pmatrix}$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\boxed{\Omega_{j,l,m}(\theta,\phi)} \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta,\phi) \end{pmatrix}$$

- Spinor spherical harmonics:

$$\Omega_{j,l,m} = Y_{l,m_l} \otimes \xi_{s,m_s}$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\boxed{\Omega_{j,l,m}(\theta,\phi)} \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta,\phi) \end{pmatrix}$$

- Spinor spherical harmonics:

$$\Omega_{j,l,m} = Y_{l,m_l} \otimes \xi_{s,m_s}$$

$$\int \Omega^\dagger_{j,l,m}\Omega_{j',l',m'} \sin\theta \, d\phi \, d\theta = \delta_{j,j'}\delta_{l,l'}\delta_{m,m'}$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\Omega_{j,l,m}(\theta,\phi) \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta,\phi) \end{pmatrix}$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\Omega_{j,l,m}(\theta,\phi) \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta,\phi) \end{pmatrix}$$

- Parity:

$$\omega = l \pm 1$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\Omega_{j,l,m}(\theta,\phi) \\ g_{n,j,l}(r)\Omega_{j,\boxed{\omega},m}(\theta,\phi) \end{pmatrix}$$

- Parity:

$$\omega = l \pm 1$$

- Ensures that this is a parity eigenstate

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} \boxed{f_{n,j,l}(r)}\Omega_{j,l,m}(\theta, \phi) \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta, \phi) \end{pmatrix}$$

The relativistic hydrogen atom

- The solution:

$$\mathcal{U} \sim \begin{pmatrix} f_{n,j,l}(r)\Omega_{j,l,m}(\theta,\phi) \\ g_{n,j,l}(r)\Omega_{j,\omega,m}(\theta,\phi) \end{pmatrix}$$

- Radial wavefunctions:

$$f, g \propto {}_1F_1(a, b, cr)e^{-cr}$$

Transition rates

- The Hamiltonian:

$$\mathcal{H}_{\text{int}} = \sum_L \left(\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e \right) \mathcal{O}_{\{\mu\},L}$$

Transition rates

- The Hamiltonian:

$$\mathcal{H}_{\rm int} = \sum_L \left(\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e \right) \mathcal{O}_{\{\mu\},L}$$

- In general:

$$\Gamma_L^{\{\mu\}} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

Transition rates

- The Hamiltonian:

$$\mathcal{H}_{\text{int}} = \sum_L \left(\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e \right) \mathcal{O}_{\{\mu\},L}$$

- For QED:

Transition rates

- The Hamiltonian:

$$\mathcal{H}_{\text{int}} = \sum_L \left(\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e\right) \mathcal{O}_{\{\mu\},L}$$

- For QED:

$$\Gamma_L^{\{\mu\}} = \gamma^\mu$$

Transition rates

- The Hamiltonian:

$$\mathcal{H}_{\text{int}} = \sum_L \left(\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e \right) \mathcal{O}_{\{\mu\},L}$$

- For QED:

$$\Gamma_L^{\{\mu\}} = \gamma^\mu \qquad\qquad \mathcal{O}_{\{\mu\},L} = eA_\mu$$

Transition rates

- The Hamiltonian:

$$\mathcal{H}_{\text{int}} = \sum_L \left(\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e\right) \mathcal{O}_{\{\mu\},L}$$

- For QED:

$$\Gamma_L^{\{\mu\}} \in \{\gamma^\mu, \sigma^{\mu\nu}, \sigma^{\mu\nu}\}$$

Transition rates

- The Hamiltonian:

$$\mathcal{H}_{\text{int}} = \sum_L \left(\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e \right) \mathcal{O}_{\{\mu\},L}$$

- For QED:

$$\Gamma_L^{\{\mu\}} \in \{\gamma^\mu, \sigma^{\mu\nu}, \sigma^{\mu\nu}\}$$

$$\mathcal{O}_{\{\mu\},L} \in \{eA_\mu, \mu_B F_{\mu\nu}, \mu_E \widetilde{F}_{\mu\nu}\}$$

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} = \langle f| \int d^3x \, \mathcal{H}_{\text{int}} |i\rangle$$

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} = \langle f| \int d^3x \, \mathcal{H}_{\text{int}} |i\rangle$$

- Factorise:

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} = \langle f | \int d^3x \, \mathcal{H}_{\rm int} | i \rangle$$

- Factorise:

$$\langle f | \bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e | i \rangle = \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i$$

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} = \langle f | \int d^3x \, \mathcal{H}_{\text{int}} | i \rangle$$

- Factorise:

$$\langle f | \bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e | i \rangle = \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i$$

$$\langle f | \mathcal{O}_{\{\mu\},L} | i \rangle = \langle \mathcal{O}_{\{\mu\},L} \rangle e^{i\vec{k}\cdot\vec{x}}$$

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} = \langle f| \int d^3x \, \mathcal{H}_{\text{int}} |i\rangle$$

- Factorise:

$$\langle f|\bar{\psi}_e \Gamma_L^{\{\mu\}} \psi_e |i\rangle = \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i$$

$$\langle f|\mathcal{O}_{\{\mu\},L}|i\rangle = \langle \mathcal{O}_{\{\mu\},L}\rangle e^{i\vec{k}\cdot\vec{x}} \qquad\qquad e^{i\vec{k}\cdot\vec{x}} \simeq 1$$

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} = \langle f| \int d^3x \, \mathcal{H}_{\text{int}} |i\rangle$$

- In QED:

$$\langle \mathcal{O}_{V,\mu} \rangle = e\langle \gamma | A_\mu | 0 \rangle = \frac{e}{\sqrt{2E_\gamma}} \epsilon^*_{h,\mu}$$

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} \simeq \sum_L \langle \mathcal{O}_{\{\mu\},L} \rangle \int d^3x \, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i$$

Transition rates

- The amplitude:

$$\mathcal{M}_{fi} \simeq \sum_{L} \langle \mathcal{O}_{\{\mu\},L} \rangle \int d^3x \, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i$$

- Seems innocuous, but squaring:

$$|\mathcal{M}_{fi}|^2 \simeq \sum_{L,L'} \langle \mathcal{O}_{\{\mu\},L} \rangle \langle \mathcal{O}_{\{\nu\},L'} \rangle^* A_{L,L'}^{\{\mu\}\{\nu\}}$$

Transition rates

- A huge mess!

Transition rates

- A huge mess!

- Up to 16 Lorentz indices per Lorentz structure

Transition rates

- A huge mess!
- Up to 16 Lorentz indices per Lorentz structure
- Potentially many Lorentz structures

Transition rates

- A huge mess!
- Up to 16 Lorentz indices per Lorentz structure
- Potentially many Lorentz structures
- Interference terms...

Transition rates

- A huge mess!

Transition rates

- A huge mess!
- Each integral is non-trivial, both analytically and numerically

Transition rates

- A huge mess!
- Each integral is non-trivial, both analytically and numerically
- What if we want to sum over polarisations?

Transition rates

- A huge mess!
- Each integral is non-trivial, both analytically and numerically
- What if we want to sum over polarisations?
- Are we doomed?

Contents

- Generalised hydrogen interactions
  - The relativistic hydrogen atom
  - Transition rates
- CINCO ⇐
- What's next?

CINCO

- Automated tool for the computation of amplitudes

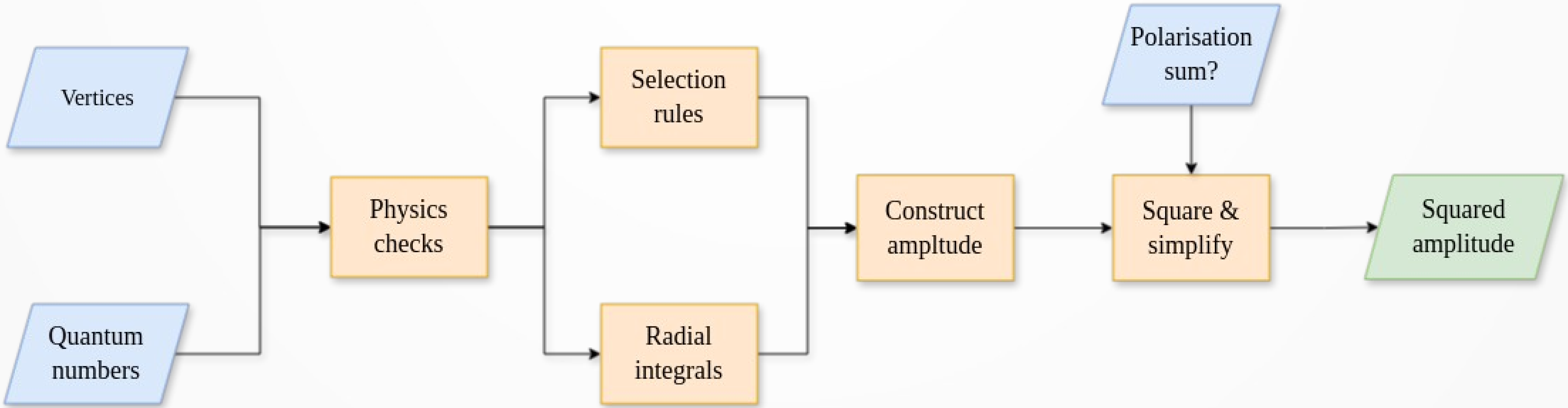
CINCO

- Automated tool for the computation of amplitudes
- We have a Hamiltonian, and a set of quantum numbers

CINCO

- Automated tool for the computation of amplitudes
- We have a Hamiltonian, and a set of quantum numbers
- Want to efficiently turn this into an amplitude

CINCO


Vertices
Quantum numbers
Physics checks
Selection rules
Radial integrals
Construct ampltude
Polarisation sum?
Square & simplify
Squared amplitude

CINCO

Vertices

CINCO

Vertices

$$\Gamma_L^{\{\mu\}} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

CINCO

Vertices

Quantum numbers

$$\Gamma_L^{\{\mu\}} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

CINCO

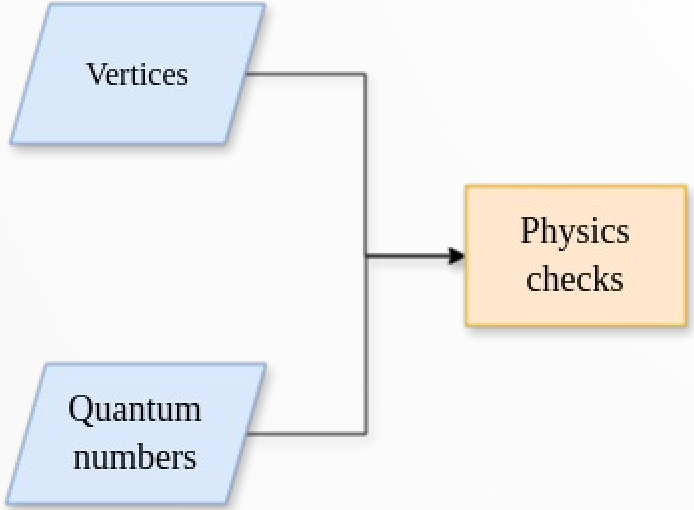
Vertices

Quantum numbers

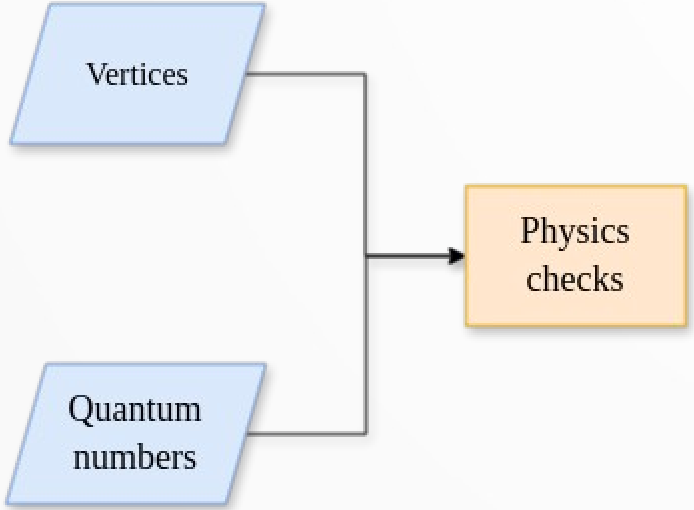
$$\Gamma_L^{\{\mu\}} \in \{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$$

$$n, n', j, j', \ldots$$

CINCO

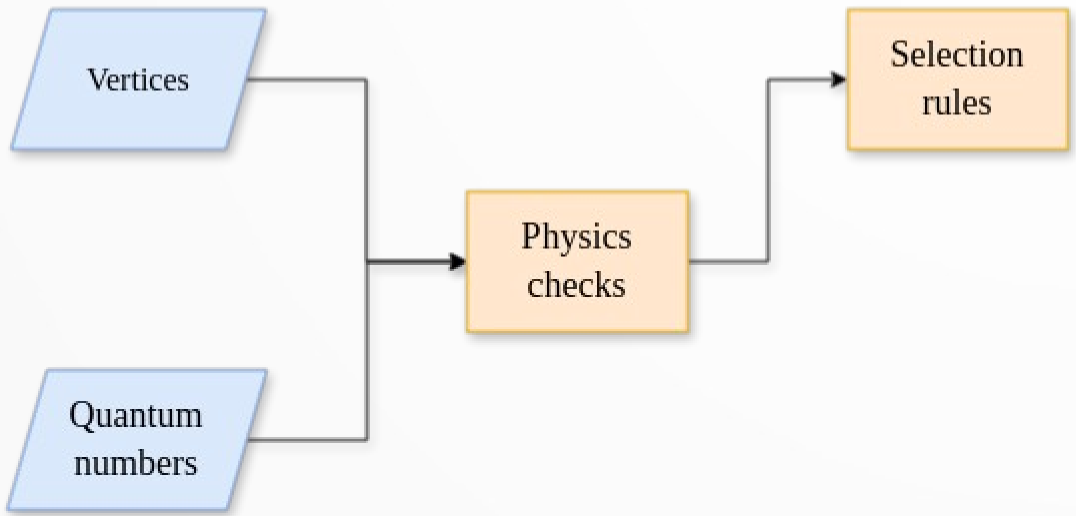


CINCO



$$l \leq n - 1, \qquad -l \leq m \leq l$$

CINCO



CINCO: Selection rules

- The squared amplitude:

$$|\mathcal{M}_{fi}|^2 \simeq \sum_{L,L'} \langle \mathcal{O}_{\{\mu\},L} \rangle \langle \mathcal{O}_{\{\nu\},L'} \rangle^* A^{\{\mu\}\{\nu\}}_{L,L'}$$

CINCO: Selection rules

- The squared amplitude:

$$|\mathcal{M}_{fi}|^2 \simeq \sum_{L,L'} \langle \mathcal{O}_{\{\mu\},L} \rangle \langle \mathcal{O}_{\{\nu\},L'} \rangle^* \boxed{A^{\{\mu\}\{\nu\}}_{L,L'}}$$

- The atomic tensor:

$$A^{\{\mu\}\{\nu\}}_{L,L'} = \left(\int d^3x \, \bar{\mathcal{U}}_f \Gamma^{\{\mu\}}_L \mathcal{U}_i \right) \left(\dots \right)^*$$

CINCO: Selection rules

- Nasty looking integral:

$$\int d^3x\, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i$$

CINCO: Selection rules

- Nasty looking integral:

$$\int d^3x\, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i \sim \mathcal{I}_\Omega \times \mathcal{I}_r$$

CINCO: Selection rules

- Nasty looking integral:

$$\int d^3x\, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i \sim \mathcal{I}_\Omega \times \mathcal{I}_r$$

- Angular integrals give us all of the selection rules:

CINCO: Selection rules

- Nasty looking integral:

$$\int d^3x\, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i \sim \mathcal{I}_\Omega \times \mathcal{I}_r$$

- Angular integrals give us all of the selection rules:

$$\boxed{\int \Omega^\dagger_{j,l,m} \Omega_{j',l',m'} \sin\theta \, d\phi \, d\theta = \delta_{j,j'} \delta_{l,l'} \delta_{m,m'}}$$

CINCO: Selection rules

- But there's an issue

CINCO: Selection rules

- But there's an issue
- We don't (obviously) have this orthogonal integral:

CINCO: Selection rules

- But there's an issue

- We don't (obviously) have this orthogonal integral:

$$\Gamma_L^{\{\mu\}} \longrightarrow \sigma_i$$

CINCO: Selection rules

- But there's an issue

- We don't (obviously) have this orthogonal integral:

$$\Gamma_L^{\{\mu\}} \longrightarrow \sigma_i$$

$$\mathcal{I}_\Omega \sim \int \Omega_f^\dagger \, \sigma \, \Omega_i \sin\theta \, d\theta \, d\phi = ?$$

CINCO: Selection rules

- So what now?

CINCO: Selection rules

- So what now?

- Want to somehow massage this and recover orthogonality:

CINCO: Selection rules

- So what now?

- Want to somehow massage this and recover orthogonality:

$$(j, l) \to \kappa$$

CINCO: Selection rules

- So what now?

- Want to somehow massage this and recover orthogonality:

$$(j, l) \to \kappa \qquad\qquad (x, y, z) \to (+, -, z)$$

CINCO: Selection rules

- What is κ ?

CINCO: Selection rules

- What is κ?

- Related to projection of spin onto orbital angular momentum

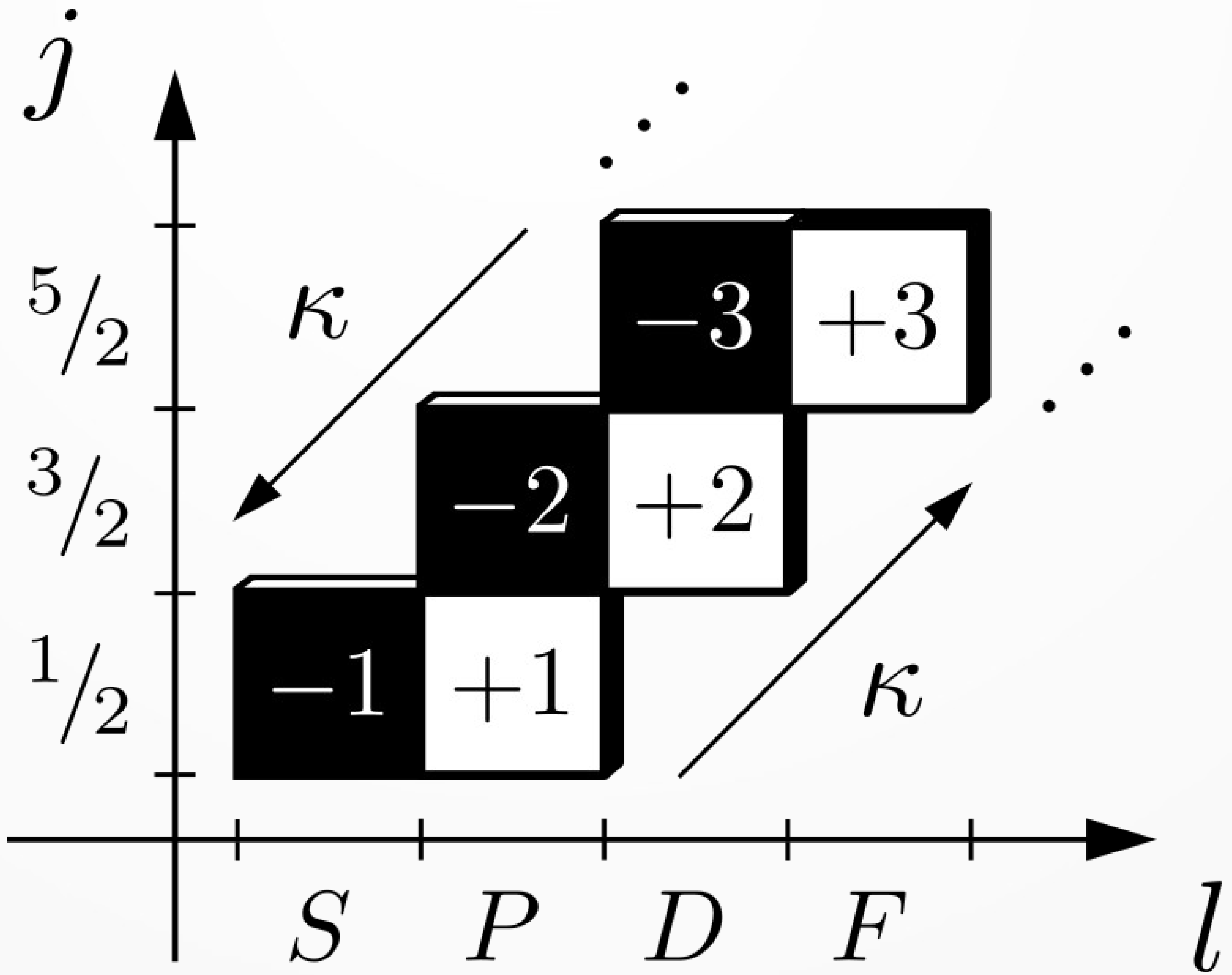
CINCO: Selection rules

- What is κ?

- Related to projection of spin onto orbital angular momentum

- A trick! Reduces two quantum numbers to one

CINCO: Selection rules



CINCO: Selection rules

- First piece of the puzzle!

$$\Omega_{j,l,m} \longrightarrow \Omega_{\kappa,m}$$

CINCO: Selection rules

- First piece of the puzzle!

$$\Omega_{j,l,m} \longrightarrow \Omega_{\kappa,m}$$

- Second piece, the spherical basis:

$$\sigma_x, \sigma_y \longrightarrow \sigma_+, \sigma_-$$

CINCO: Selection rules

- Why are we doing all of this?

CINCO: Selection rules

- Why are we doing all of this?

- Recurrence relations in spherical basis [1]:

$$\sigma_{\pm,z}\Omega_{\kappa,m} = \sum_i C_i \Omega_{\kappa',m'}$$

[1] R. Szmytkowski, J.Math.Chem. **42,** (2007) 397-413

CINCO: Selection rules

- Why are we doing all of this?

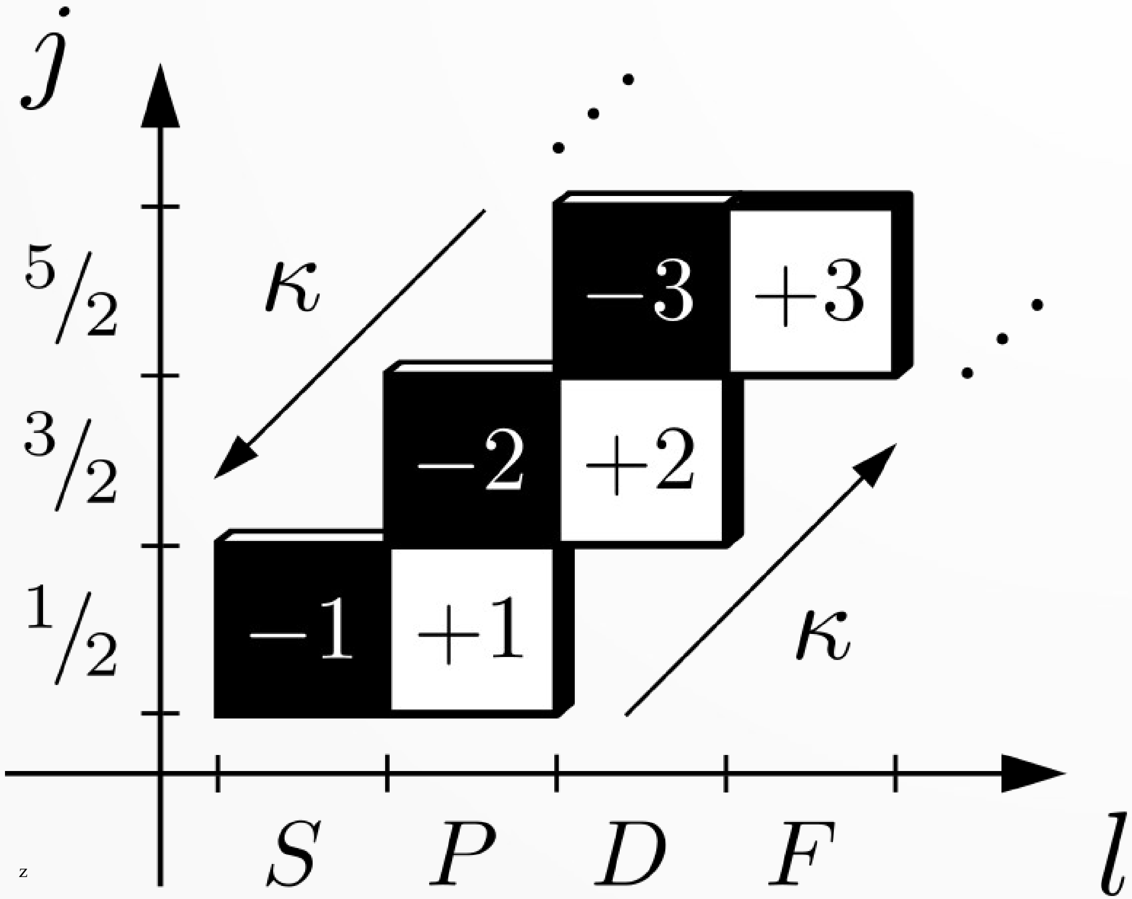
- Recurrence relations in spherical basis [1]:

$$\sigma_{\pm,z}\Omega_{\kappa,m} = \sum_i C_i \Omega_{\kappa',m'}$$

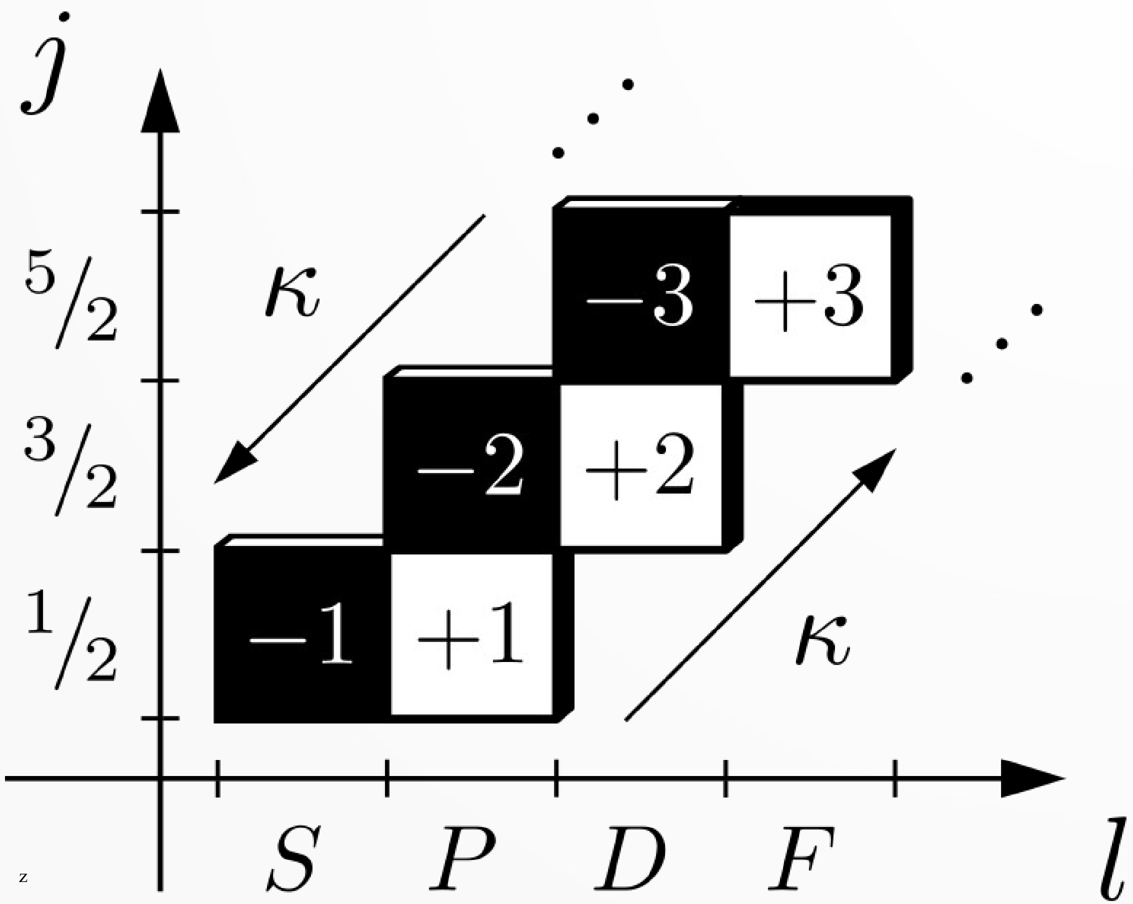
- Recovers orthogonality!

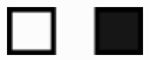
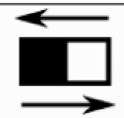
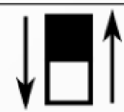
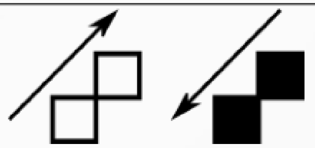
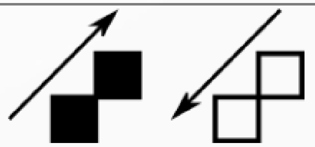
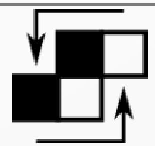
[1] R. Szmytkowski, J.Math.Chem. **42,** (2007) 397-413

CINCO: Selection rules

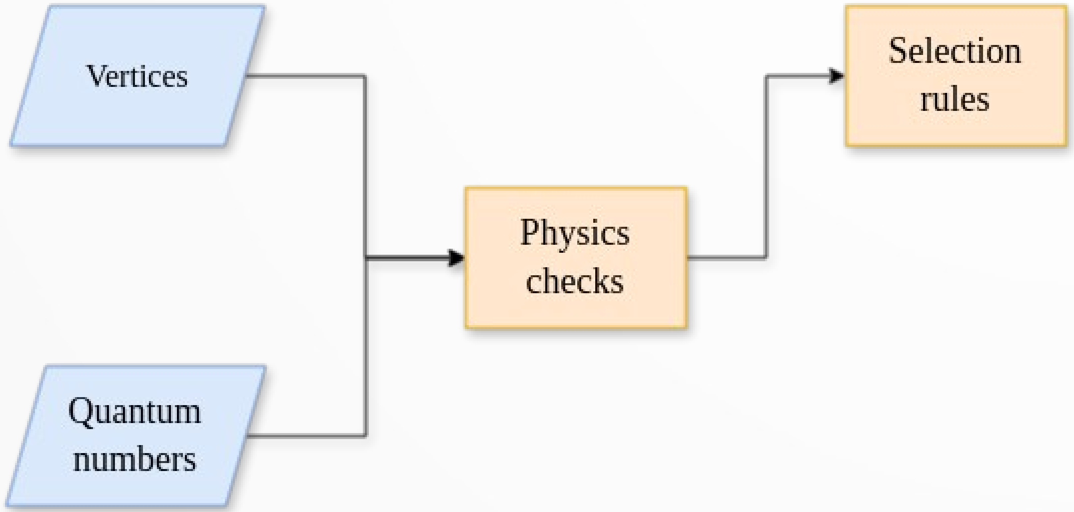


CINCO: Selection rules



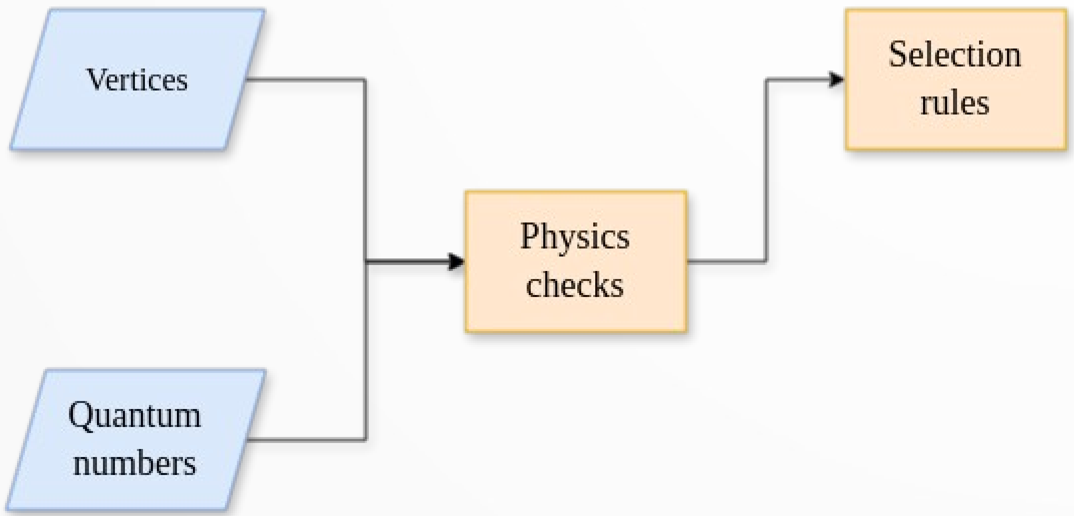
$\kappa \rightarrow \ldots$	Transitions	Couplings
κ		S, A, T
$-\kappa$		P, V, A, T
$-\kappa-1$		A, T
$\kappa+1$		V, T
$\kappa-1$		V, T
$-\kappa+1$		A, T

CINCO



$$\int d^3x\, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i \sim \mathcal{I}_\Omega \times \mathcal{I}_r$$

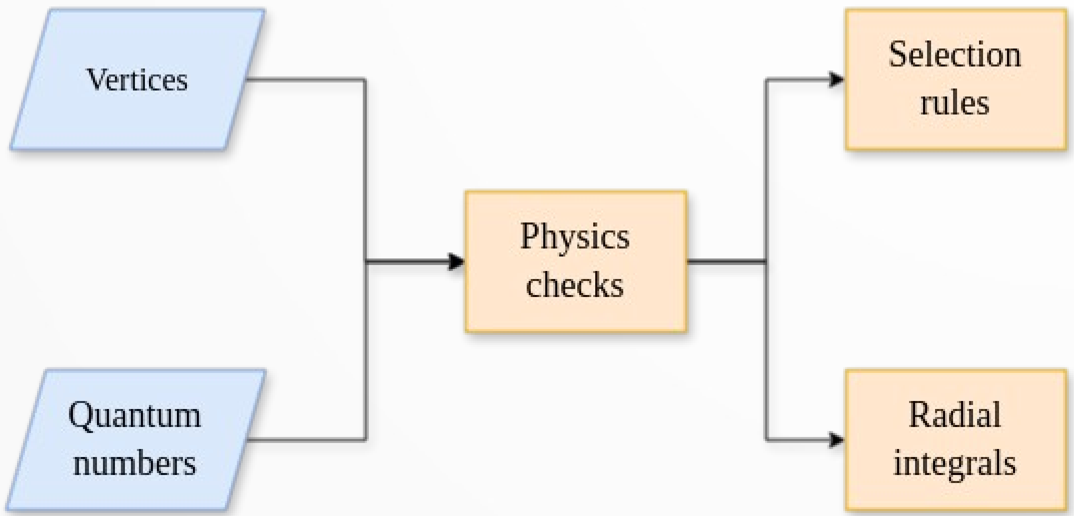
CINCO



$$\int d^3x\, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i \sim \mathcal{I}_\Omega \times \mathcal{I}_r$$

$$\mathcal{I}_\Omega \sim \sum_i C_i \delta_i$$

CINCO



$$\int d^3x \, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i \sim \mathcal{I}_\Omega \times \mathcal{I}_r$$

$$\mathcal{I}_\Omega \sim \sum_i C_i \delta_i$$

CINCO: Radial integrals

- Products of radial wavefunctions:

$$\mathcal{I}_r \sim \int_0^\infty r^2 f_{n',\kappa'}(r) g_{n,\kappa}(r)\, dr$$

CINCO: Radial integrals

- Products of radial wavefunctions:

$$\mathcal{I}_r \sim \int_0^\infty r^2 f_{n',\kappa'}(r) g_{n,\kappa}(r)\, dr$$

- These are super ugly, can technically be done analytically

CINCO: Radial integrals

- Products of radial wavefunctions:

$$\mathcal{I}_r \sim \int_0^\infty r^2 f_{n',\kappa'}(r) g_{n,\kappa}(r)\, dr$$

- These are super ugly, can technically be done analytically
- No real benefit to doing so

CINCO: Radial integrals

- Solution? Do it numerically!

CINCO: Radial integrals

- Solution? Do it numerically!

- But there's an issue:

$$f, g \propto {}_1F_1(a, b, cr)e^{-cr}$$

CINCO: Radial integrals

- Solution? Do it numerically!

- But there's an issue:

$$f, g \propto {}_1F_1(a, b, cr)e^{-cr}$$

$$\lim_{r \to \infty} {}_1F_1(a, b, cr) \to \infty$$

CINCO: Radial integrals

- Solution? Do it numerically!

- But there's an issue:

$$f, g \propto {}_1F_1(a, b, cr)e^{-cr}$$

$$\lim_{r \to \infty} {}_1F_1(a, b, cr) \to \infty \qquad \lim_{r \to \infty} e^{-cr} = 0$$

CINCO: Radial integrals

- Solution? Do it numerically!

- But there's an issue:

$$f, g \propto {}_1F_1(a, b, cr)e^{-cr}$$

$$\lim_{r \to \infty} {}_1F_1(a, b, cr) \to \infty \qquad \lim_{r \to \infty} e^{-cr} = 0$$

- Arbitrary floating point precision!

CINCO: Radial integrals

- Four integrals:

CINCO: Radial integrals

- Four integrals:

$$\mathcal{I}_{ff} \sim Z^0 \qquad \mathcal{I}_{fg}, \mathcal{I}_{gf} \sim Z^1 \qquad \mathcal{I}_{gg} \sim Z^2$$

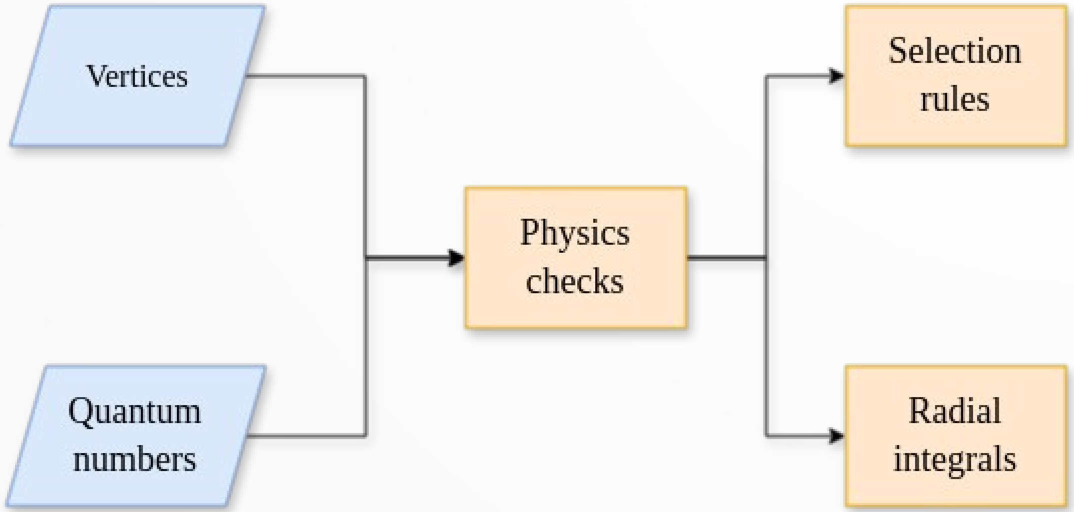
CINCO: Radial integrals

- Four integrals:

$$\mathcal{I}_{ff} \sim Z^0 \qquad \mathcal{I}_{fg}, \mathcal{I}_{gf} \sim Z^1 \qquad \mathcal{I}_{gg} \sim Z^2$$

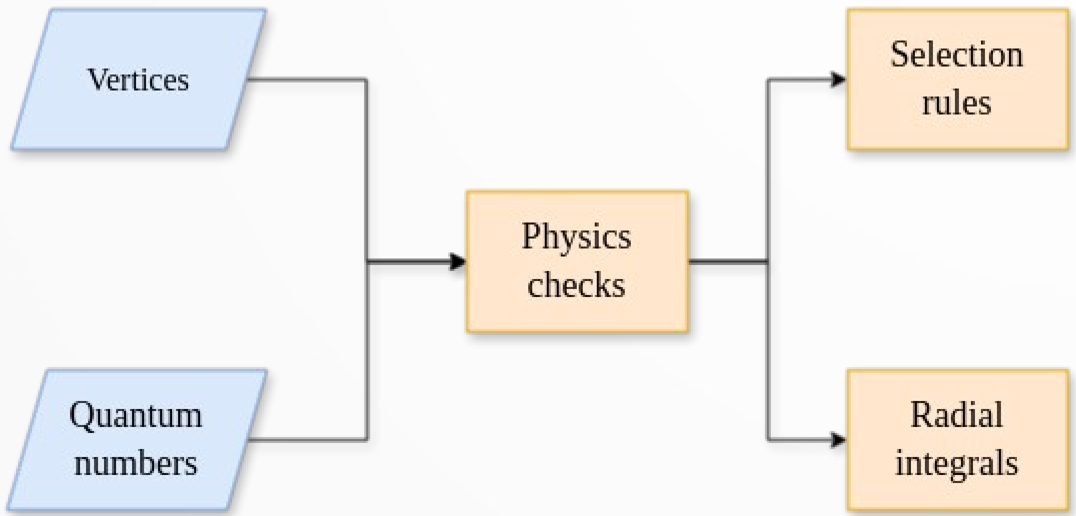
- Contribute to relativistic scaling

CINCO



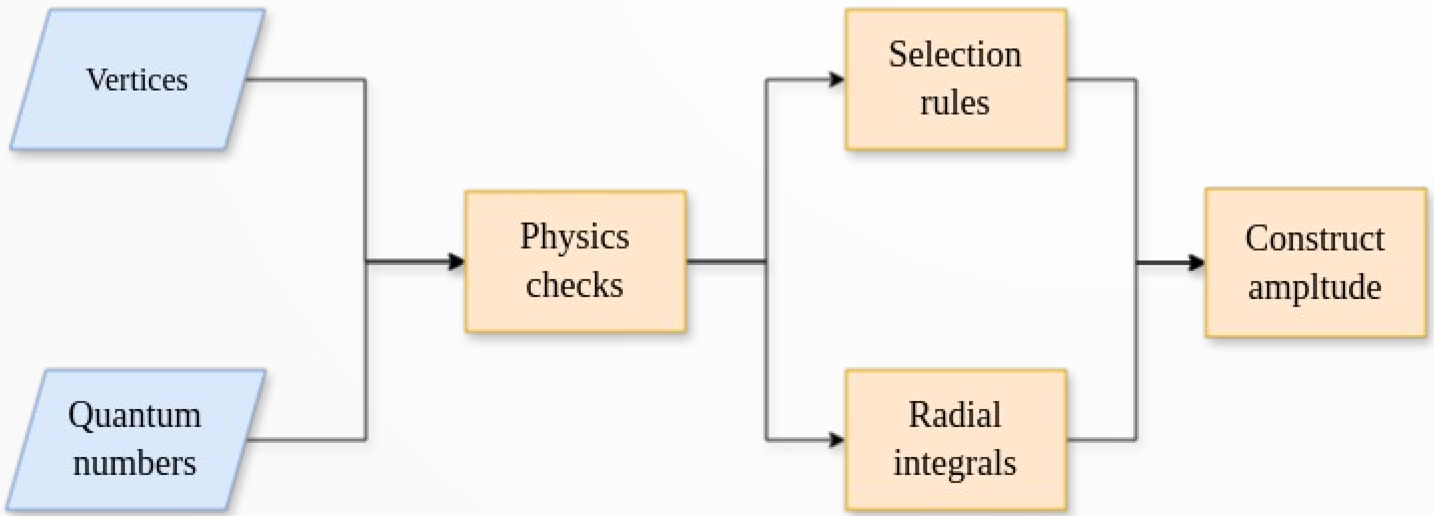
$$\int d^3x\, \bar{\mathcal{U}}_f \Gamma_L^{\{\mu\}} \mathcal{U}_i \sim \mathcal{I}_\Omega \times \mathcal{I}_r$$

CINCO



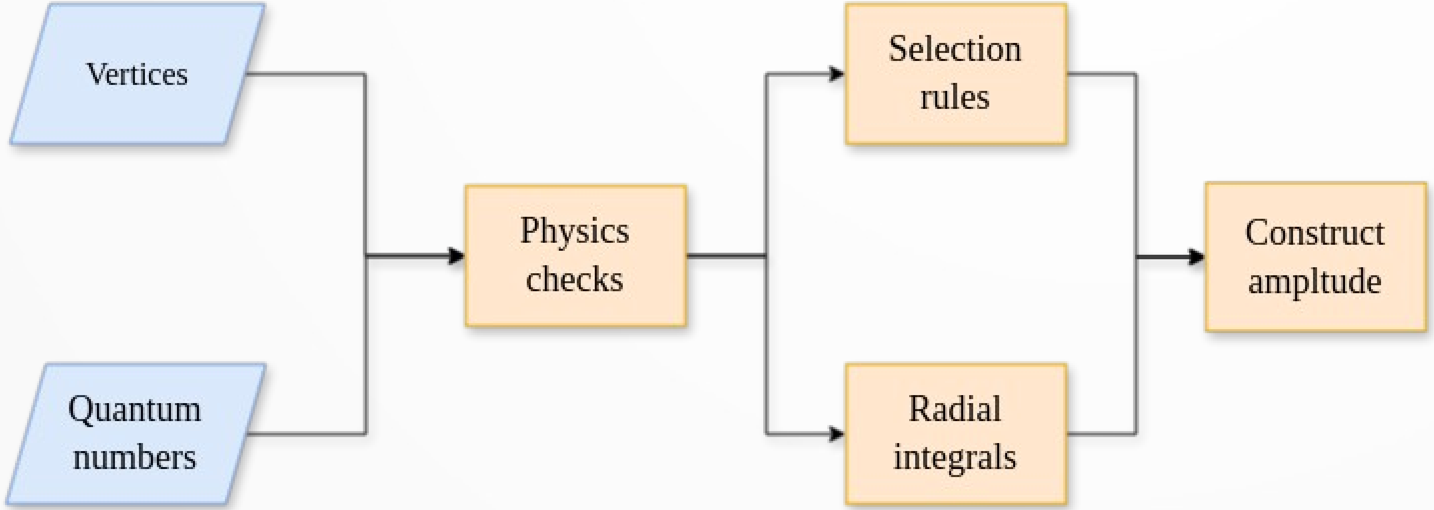
$$\mathcal{I}_\Omega \sim \sum_i C_i \delta_i \qquad \mathcal{I}_r \in \{\mathcal{I}_{ff}, \mathcal{I}_{fg}, \ldots\}$$

CINCO



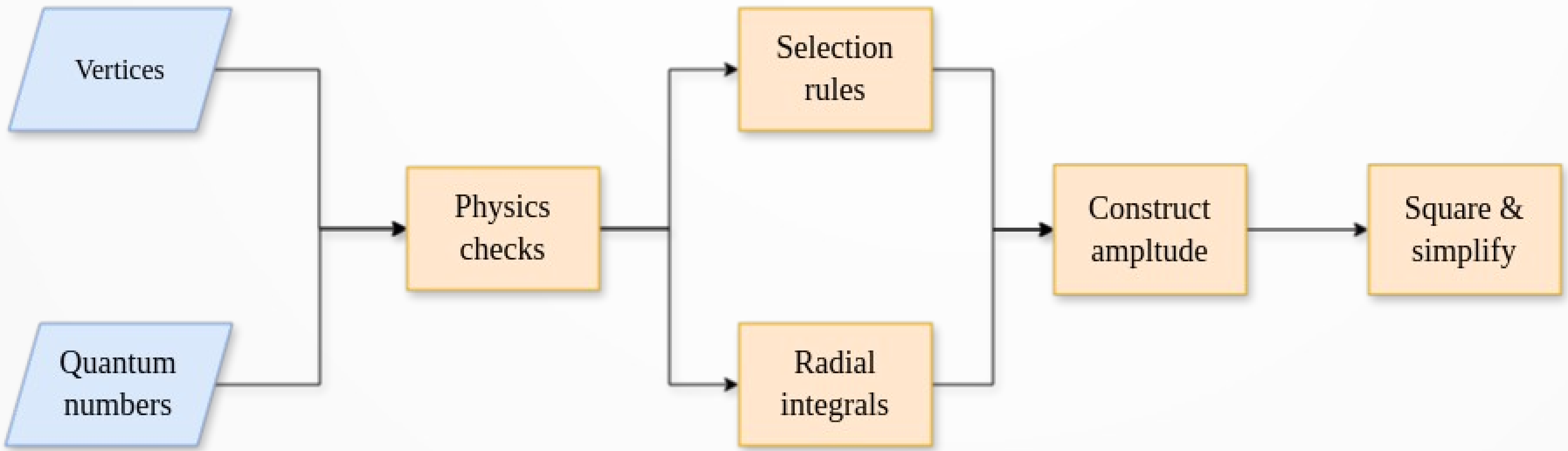
$$|\mathcal{M}_{fi}|^2 \simeq \sum_{L,L'} \langle \mathcal{O}_{\{\mu\},L} \rangle \langle \mathcal{O}_{\{\nu\},L'} \rangle^* A_{L,L'}^{\{\mu\}\{\nu\}}$$

CINCO

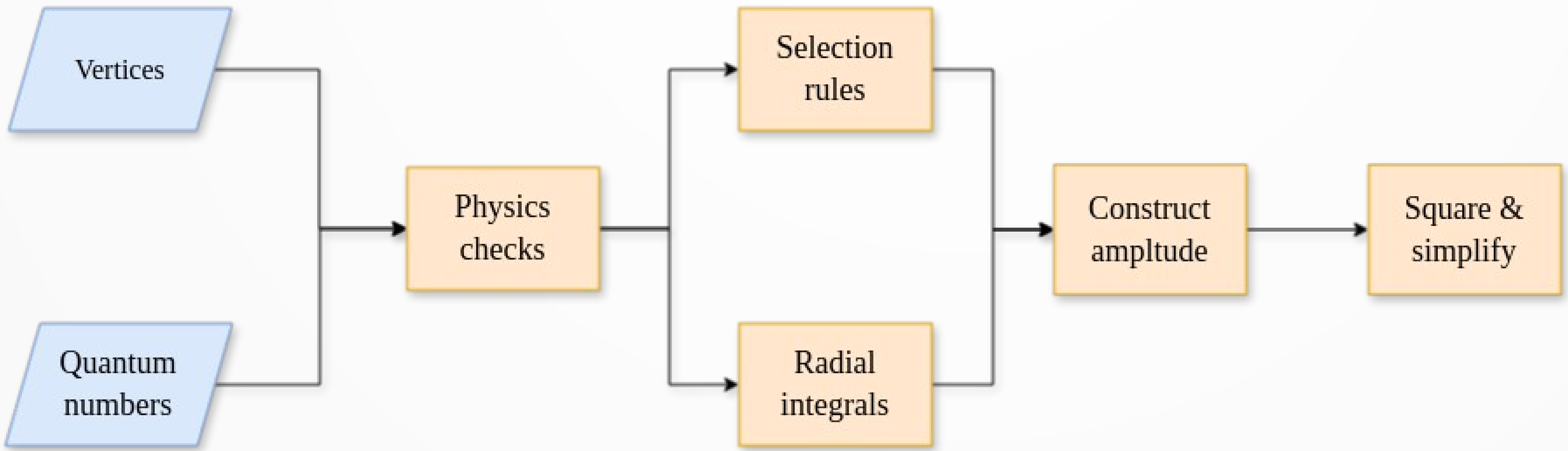


$$A_{L,L'}^{\{\mu\}\{\nu\}} \to \sum_i C_i \delta_i \mathcal{I}_{r,i}$$

CINCO

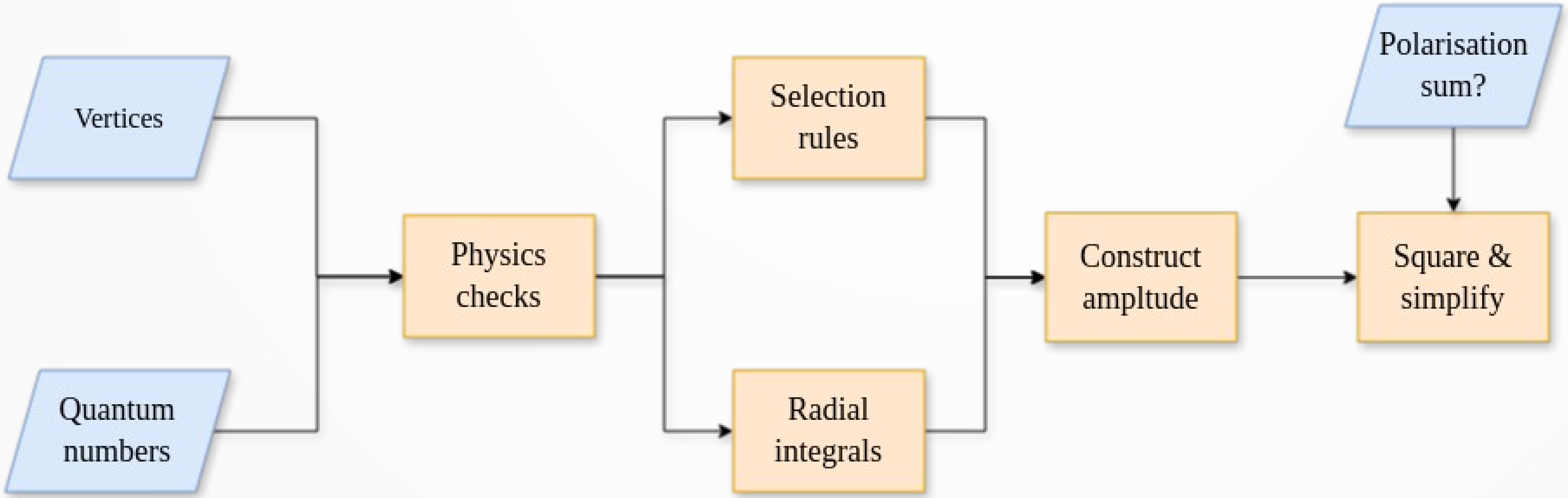

Vertices
Quantum numbers
Physics checks
Selection rules
Radial integrals
Construct ampltude
Square & simplify

CINCO

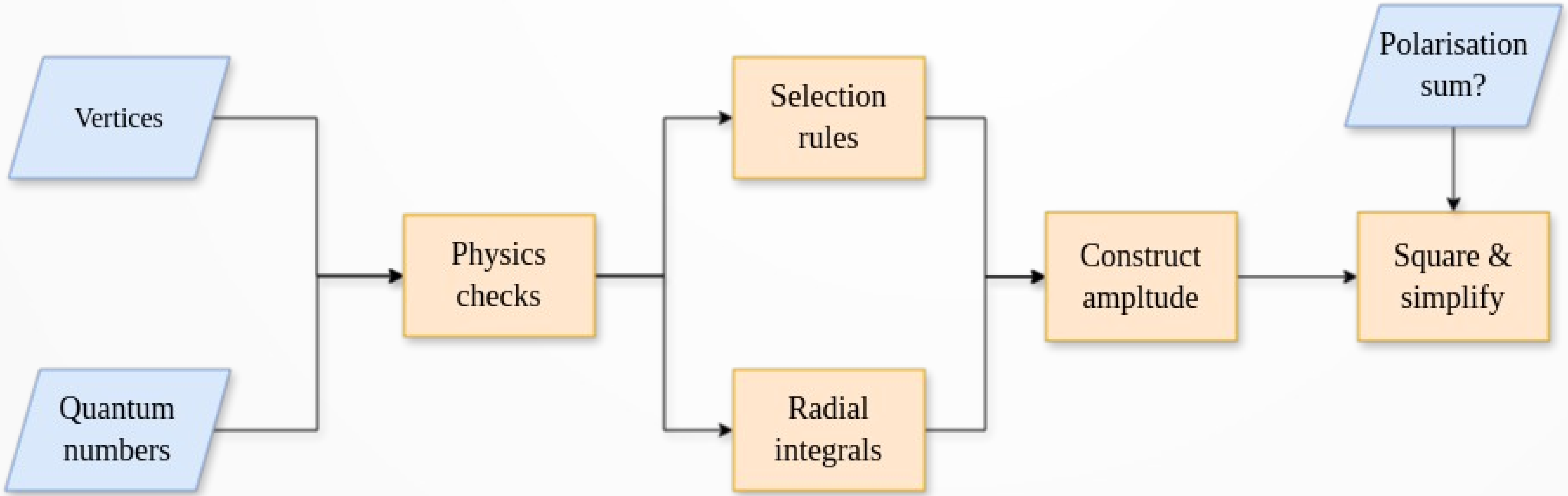


$$\delta_i \delta_j \rightarrow 0$$

CINCO

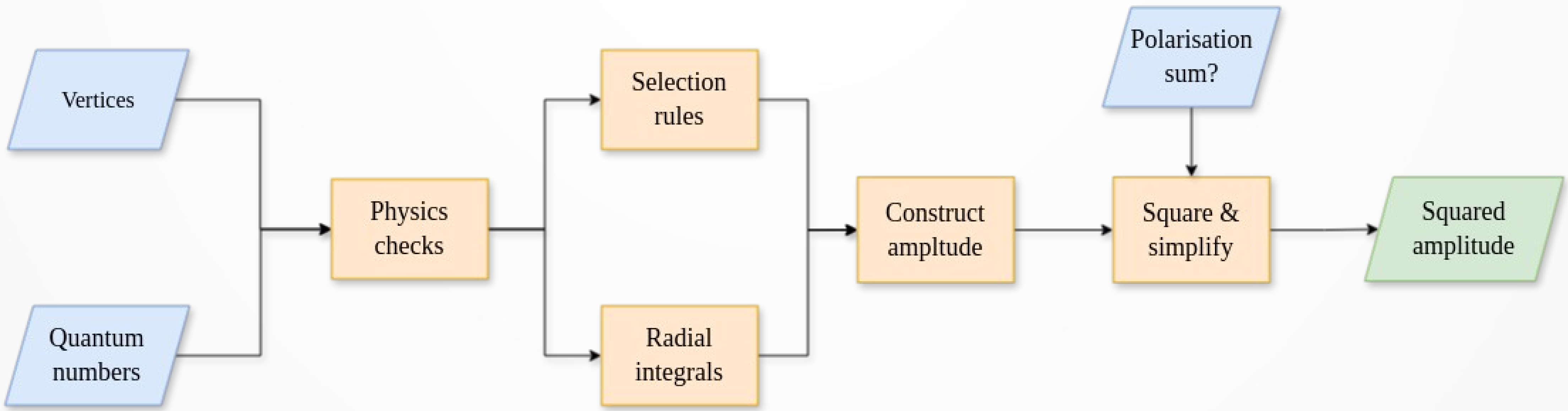

Vertices
Quantum numbers
Physics checks
Selection rules
Radial integrals
Construct ampltude
Polarisation sum?
Square & simplify

CINCO



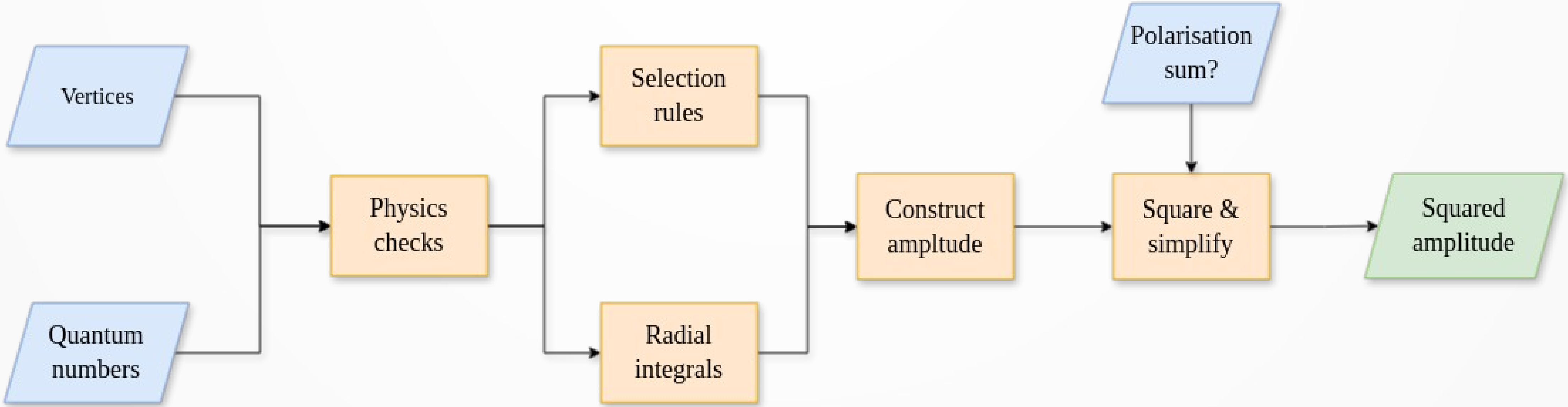
$$|\mathcal{M}_{fi}|^2 \rightarrow \langle|\mathcal{M}_{fi}|^2\rangle_{m,m'} = \frac{1}{2|\kappa|}\sum_{m,m'}|\mathcal{M}_{fi}|^2$$

CINCO



Vertices
Quantum numbers
Physics checks
Selection rules
Radial integrals
Construct ampltude
Polarisation sum?
Square & simplify
Squared amplitude

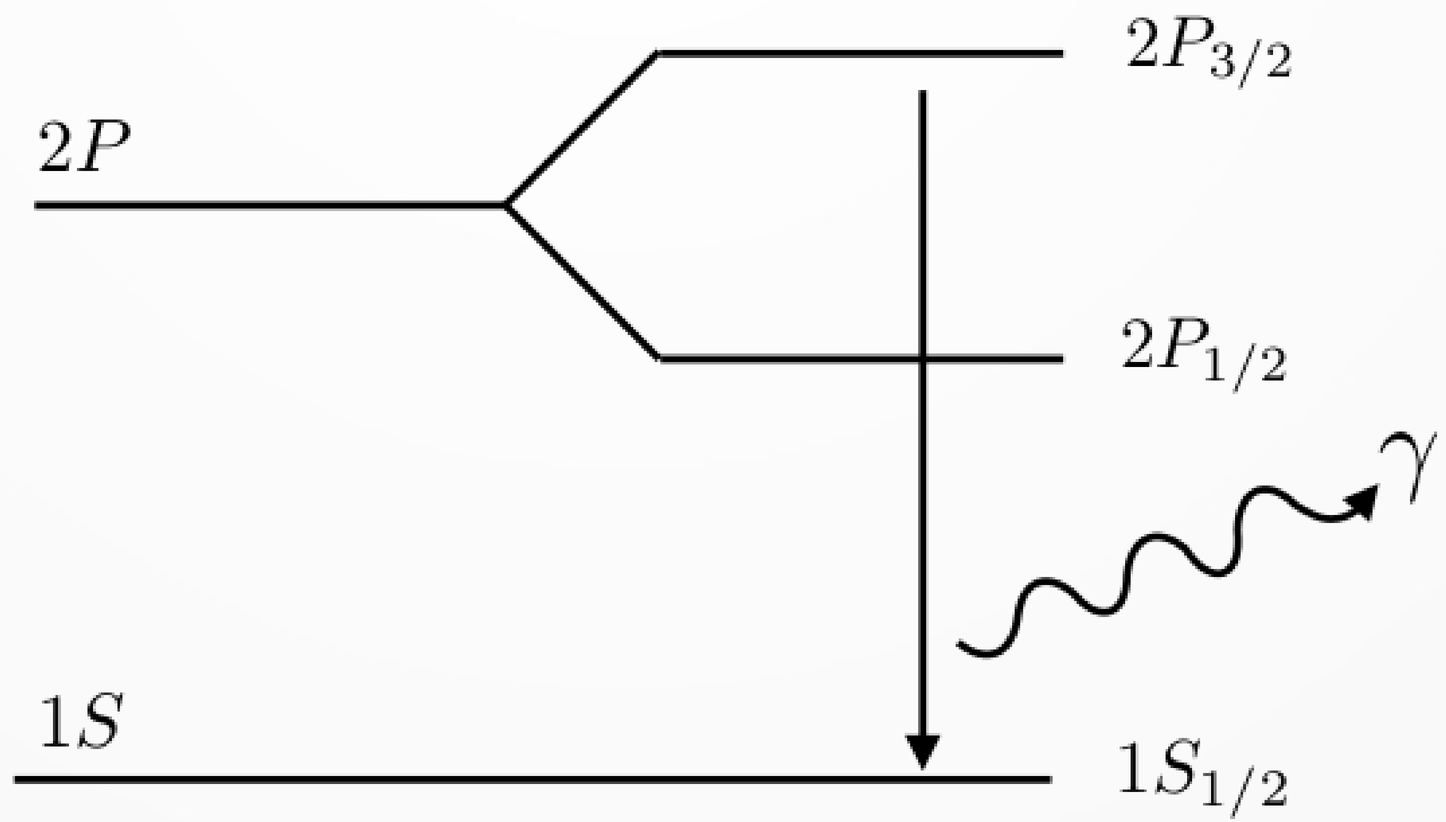
CINCO



- Done! Takes around 20 seconds per amplitude

CINCO: Examples

- QED:



CINCO: Examples

CINCO: Examples

$$\left(n = 2, j = \frac{3}{2}, l = 1\right)$$

CINCO

CINCO: Examples

$$\left(n = 2, j = \frac{3}{2}, l = 1\right)$$

$$\left(n' = 1, j' = \frac{1}{2}, l' = 0\right)$$

CINCO

CINCO: Examples

$$\left(n = 2, j = \frac{3}{2}, l = 1\right)$$

$$\left(n' = 1, j' = \frac{1}{2}, l' = 0\right)$$

$$\left(\Gamma_L^{\{\mu\}} = \gamma^\mu\right)$$

CINCO

CINCO: Examples

$$\left(n = 2, j = \frac{3}{2}, l = 1\right)$$

$$\left(n' = 1, j' = \frac{1}{2}, l' = 0\right)$$

$$\left(\Gamma_L^{\{\mu\}} = \gamma^\mu\right)$$

CINCO

$$\langle |\mathcal{M}_{fi}|^2 \rangle_{m,m'}$$

CINCO: Examples

$$\left(n = 2, j = \frac{3}{2}, l = 1\right)$$

$$\left(n' = 1, j' = \frac{1}{2}, l' = 0\right)$$

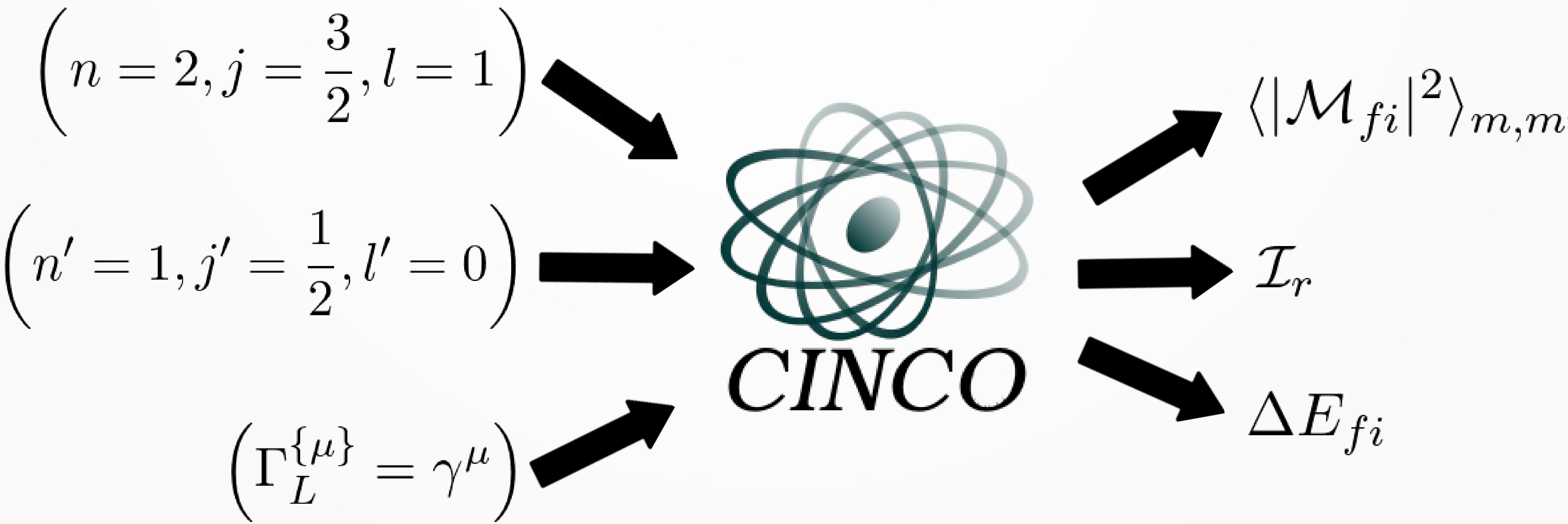
$$\left(\Gamma_L^{\{\mu\}} = \gamma^\mu\right)$$

CINCO

$$\langle |\mathcal{M}_{fi}|^2 \rangle_{m,m'}$$

$$\mathcal{I}_r$$

CINCO: Examples



CINCO: Examples

- The amplitude:

$$\langle |\mathcal{M}_{fi}|^2 \rangle_{m,m'} = \frac{4}{9} \left(\langle \vec{\mathcal{O}}_V \rangle \cdot \langle \vec{\mathcal{O}}_V \rangle^* \right) \mathcal{I}_{gf}^2$$

CINCO: Examples

- The amplitude:

$$\langle |\mathcal{M}_{fi}|^2 \rangle_{m,m'} = \frac{4}{9} \left(\langle \vec{\mathcal{O}}_V \rangle \cdot \langle \vec{\mathcal{O}}_V \rangle^* \right) \mathcal{I}_{gf}^2$$

- The radial integral and transition energy:

$$\mathcal{I}_{gf} = -0.0017654, \qquad \Delta E_{fi} = 10.199\,\mathrm{eV}$$

CINCO: Examples

- The amplitude:

$$\langle |\mathcal{M}_{fi}|^2 \rangle_{m,m'} = \frac{4}{9} \left(\langle \vec{\mathcal{O}}_V \rangle \cdot \langle \vec{\mathcal{O}}_V \rangle^* \right) \mathcal{I}_{gf}^2$$

- From before (not done by CINCO):

$$\langle \mathcal{O}_{V,\mu} \rangle = e \langle \gamma | A_\mu | 0 \rangle = \frac{e}{\sqrt{2E_\gamma}} \epsilon_{h,\mu}^*$$

CINCO: Examples

- The amplitude:

$$\langle |\mathcal{M}_{fi}|^2 \rangle_{m,m'} = \frac{8\pi\alpha_{\text{EM}}}{9E_\gamma} \left(\vec{\epsilon}_h \cdot \vec{\epsilon}_h^*\right) \mathcal{I}_{gf}^2$$

CINCO: Examples

- Fermi's golden rule, phase space integral:

$$\Gamma_{2S_{3/2}\to 1S_{1/2}} = \frac{16\alpha_{\text{EM}}}{9}\Delta E_{fi}\mathcal{I}_{gf}^2$$

CINCO: Examples

- Fermi's golden rule, phase space integral:

$$\Gamma_{2S_{3/2}\to 1S_{1/2}} = 6.2650 \cdot 10^8\,\mathrm{s}^{-1}$$

CINCO: Examples

- Fermi's golden rule, phase space integral:

$$\Gamma_{2S_{3/2}\to 1S_{1/2}} = 6.2650 \cdot 10^8\,\mathrm{s}^{-1}$$

- From NIST:

$$\Gamma_{2S_{3/2}\to 1S_{1/2}} = 6.2647 \cdot 10^8\,\mathrm{s}^{-1}$$

CINCO: Examples

- Fermi's golden rule, phase space integral:

$$\Gamma_{2S_{3/2}\to 1S_{1/2}} = 6.2650 \cdot 10^8\,\mathrm{s}^{-1}$$

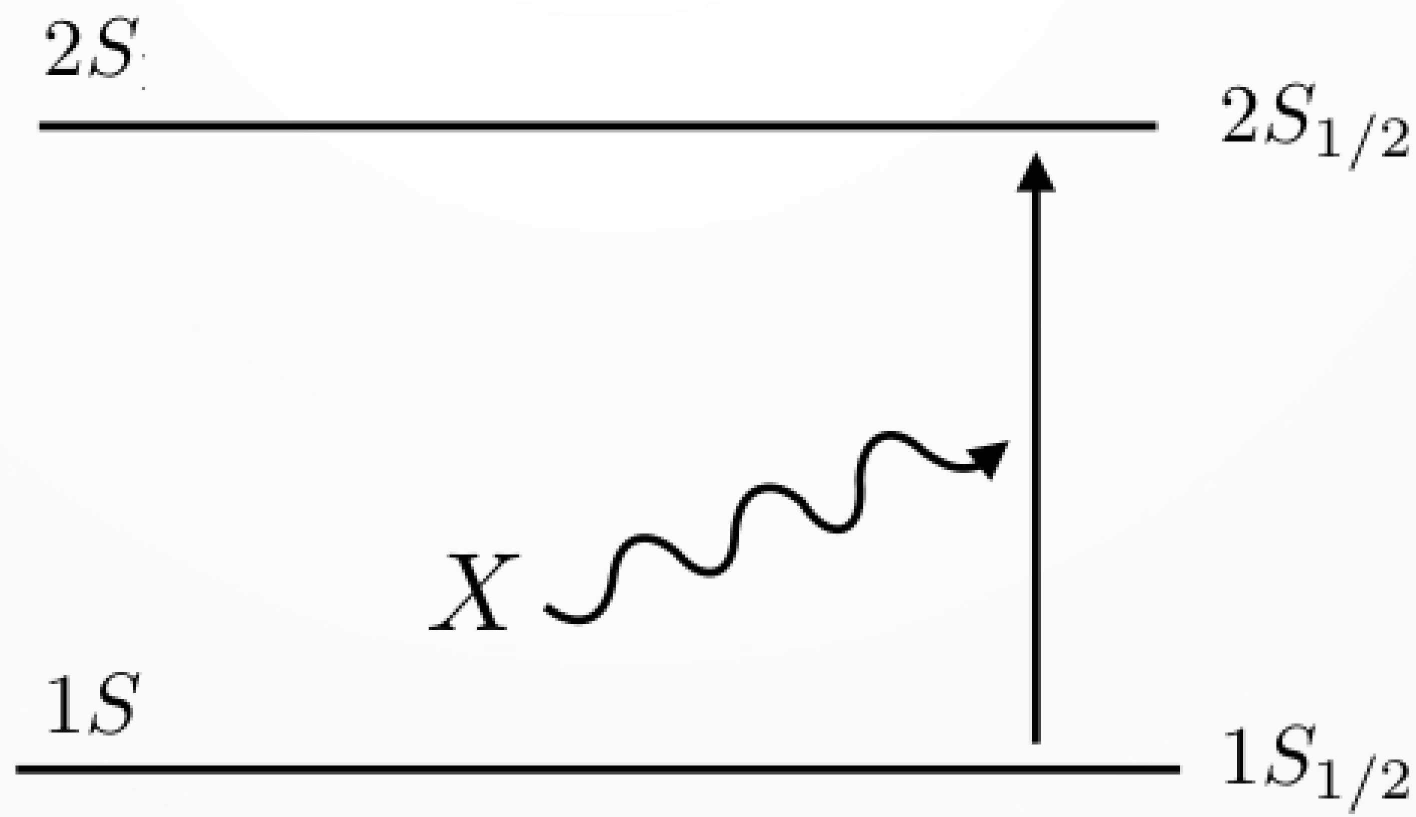
- From NIST:

$$\Gamma_{2S_{3/2}\to 1S_{1/2}} = 6.2647 \cdot 10^8\,\mathrm{s}^{-1}$$

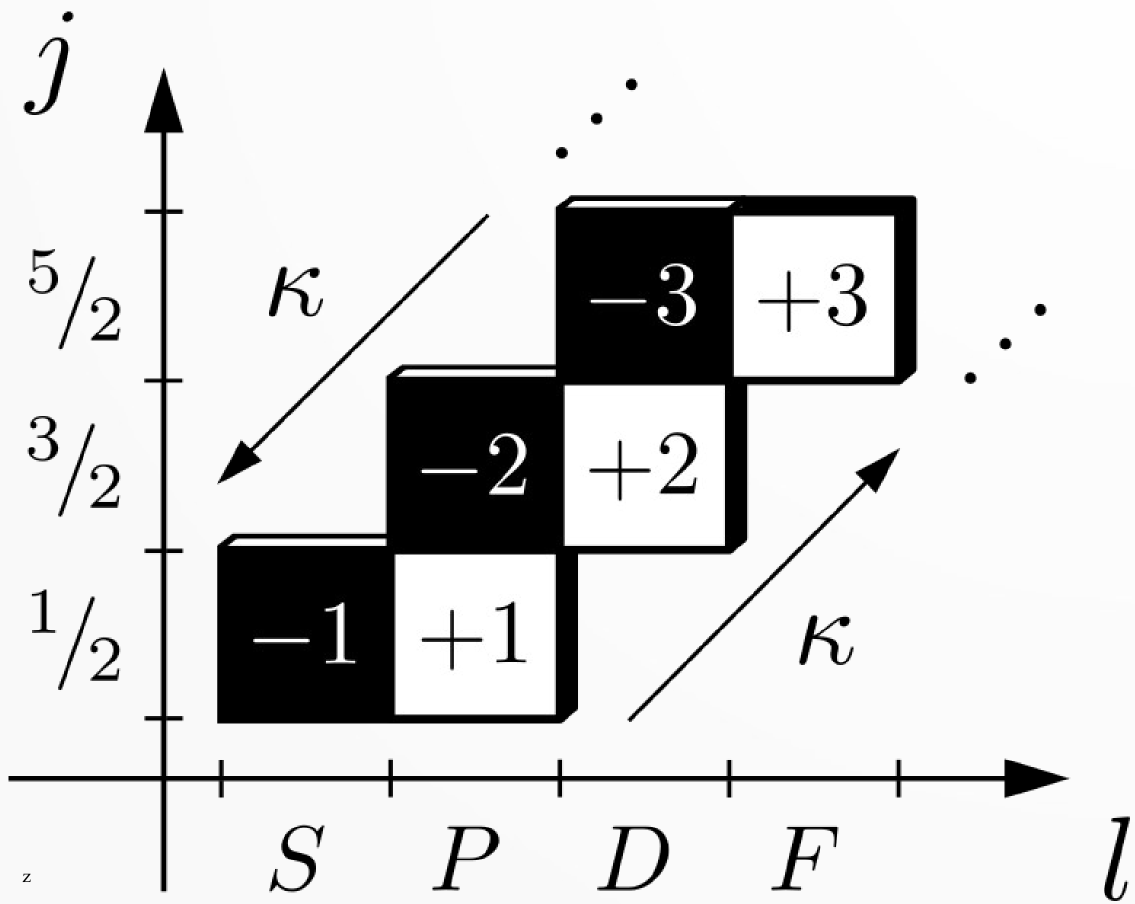
- Agrees to within 0.005%!

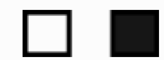
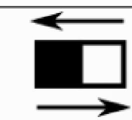
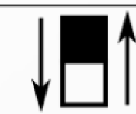
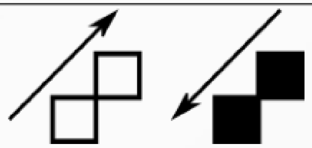
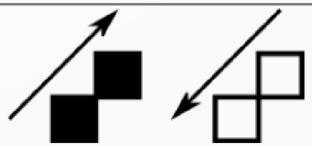
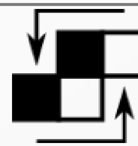
CINCO: Examples

- Dark vector boson:



CINCO: Selection rules



$\kappa \rightarrow \dots$	Transitions	Couplings
κ		S, A, T
$-\kappa$		P, V, A, T
$-\kappa - 1$		A, T
$\kappa + 1$		V, T
$\kappa - 1$		V, T
$-\kappa + 1$		A, T

CINCO: Examples

- Dark vector boson:

$$\mathcal{H}_{\text{int}} \sim X_\mu \bar{\psi}_e \gamma^\mu (1 \pm \gamma^5) \psi_e$$

- Many more terms to compute

- No longer feasible by hand

CINCO: Examples

CINCO: Examples

$$\left(n = 1, j = \frac{1}{2}, l = 0\right)$$

CINCO

CINCO: Examples

$$\left(n = 1, j = \frac{1}{2}, l = 0\right)$$

$$\left(n' = 2, j' = \frac{1}{2}, l' = 1\right)$$

CINCO

CINCO: Examples

$$\left(n = 1, j = \frac{1}{2}, l = 0\right)$$

$$\left(n' = 2, j' = \frac{1}{2}, l' = 1\right)$$

$$\left(\Gamma_L^{\{\mu\}} = \gamma^\mu, \gamma^\mu\gamma^5\right)$$

CINCO

CINCO: Examples

$$\left(n = 1, j = \frac{1}{2}, l = 0\right) \rightarrow$$

$$\left(n' = 2, j' = \frac{1}{2}, l' = 1\right) \rightarrow$$

$$\left(\Gamma_L^{\{\mu\}} = \gamma^\mu, \gamma^\mu\gamma^5\right) \rightarrow$$

CINCO

$$\rightarrow \langle|\mathcal{M}_{fi}|^2\rangle_{m,m'}$$

$$\rightarrow \mathcal{I}_r$$

$$\rightarrow \Delta E_{fi}$$

CINCO: Examples

- The amplitude:

CINCO: Examples

- The amplitude:

$$\langle|\mathcal{M}_{fi}|^2\rangle_{m,m'} = \left(\langle\vec{\mathcal{O}}_A\rangle\cdot\langle\vec{\mathcal{O}}_A\rangle^*\right)\left(\mathcal{I}_{ff}^2 - \frac{2}{3}\mathcal{I}_{ff}\mathcal{I}_{gg} + \frac{\mathcal{I}_{gg}^2}{9}\right)$$
$$+ \left|\langle\mathcal{O}_{A,0}\rangle\right|^2\left(2\mathcal{I}_{fg}^2 - 4\mathcal{I}_{fg}\mathcal{I}_{gf} + 2\mathcal{I}_{gf}^2\right)$$
$$+ \operatorname{Im}\left(\langle\mathcal{O}_{A,0}\rangle\langle\mathcal{O}_{A,x}\rangle^*\right)\left(2\mathcal{I}_{ff}\mathcal{I}_{gf} - 2\mathcal{I}_{ff}\mathcal{I}_{fg} + \frac{2}{3}\mathcal{I}_{fg}\mathcal{I}_{gg} - \frac{2}{3}\mathcal{I}_{gf}\mathcal{I}_{gg}\right)$$

CINCO: Examples

- The amplitude:

$$\langle|\mathcal{M}_{fi}|^2\rangle_{m,m'} = \left(\langle\vec{\mathcal{O}}_A\rangle\cdot\langle\vec{\mathcal{O}}_A\rangle^*\right)\left(\mathcal{I}_{ff}^2 - \frac{2}{3}\mathcal{I}_{ff}\mathcal{I}_{gg} + \frac{\mathcal{I}_{gg}^2}{9}\right)$$
$$+ \left|\langle\mathcal{O}_{A,0}\rangle\right|^2\left(2\mathcal{I}_{fg}^2 - 4\mathcal{I}_{fg}\mathcal{I}_{gf} + 2\mathcal{I}_{gf}^2\right)$$
$$+ \mathrm{Im}\left(\langle\mathcal{O}_{A,0}\rangle\langle\mathcal{O}_{A,x}\rangle^*\right)\left(2\mathcal{I}_{ff}\mathcal{I}_{gf} - 2\mathcal{I}_{ff}\mathcal{I}_{fg} + \frac{2}{3}\mathcal{I}_{fg}\mathcal{I}_{gg} - \frac{2}{3}\mathcal{I}_{gf}\mathcal{I}_{gg}\right)$$

- Ugly? Yes, but we got there in seconds

CINCO: Examples

- The amplitude:

$$\langle |\mathcal{M}_{fi}|^2 \rangle_{m,m'} = \left(\langle \vec{\mathcal{O}}_A \rangle \cdot \langle \vec{\mathcal{O}}_A \rangle^*\right) \left(\mathcal{I}_{ff}^2 - \frac{2}{3}\mathcal{I}_{ff}\mathcal{I}_{gg} + \frac{\mathcal{I}_{gg}^2}{9}\right)$$
$$+ \left|\langle \mathcal{O}_{A,0} \rangle\right|^2 \left(2\mathcal{I}_{fg}^2 - 4\mathcal{I}_{fg}\mathcal{I}_{gf} + 2\mathcal{I}_{gf}^2\right)$$
$$+ \operatorname{Im}\left(\langle \mathcal{O}_{A,0} \rangle \langle \mathcal{O}_{A,x} \rangle^*\right) \left(2\mathcal{I}_{ff}\mathcal{I}_{gf} - 2\mathcal{I}_{ff}\mathcal{I}_{fg} + \frac{2}{3}\mathcal{I}_{fg}\mathcal{I}_{gg} - \frac{2}{3}\mathcal{I}_{gf}\mathcal{I}_{gg}\right)$$

- Ugly? Yes, but we got there in seconds

- Only need to evaluate $\langle \mathcal{O}_A \rangle$

CINCO: Examples

- Fermi's golden rule, phase space integral

$$\Gamma_{1S_{1/2}\to 2S_{1/2}} \simeq 10\, g^2\, s^{-1}$$

CINCO: Examples

- Fermi's golden rule, phase space integral

$$\Gamma_{1S_{1/2}\to 2S_{1/2}} \simeq 10\, g^2\, s^{-1}$$

- Very quickly understand feasibility

Contents

- Generalised hydrogen interactions
  - The relativistic hydrogen atom
  - Transition rates
- CINCO
- What's next? ⇦

What’s next?

What's next?

- Precision:
  -
  -
  -

What's next?

- Precision:
  -
  -
  -
- Applications:
  -
  -
  -

What's next?

- Precision:
  - Beyond the dipole approximation
  -
  -
- Applications:
  -
  -
  -

What's next?

- Precision:
  - Beyond the dipole approximation
  - Hyperfine transitions
  -
- Applications:
  -
  -
  -

What's next?

- Precision:
  - Beyond the dipole approximation
  - Hyperfine transitions
  - Long range forces?
- Applications:
  -
  -
  -

What's next?

- Precision:
  - Beyond the dipole approximation
  - Hyperfine transitions
  - Long range forces?
- Applications:
  - Dark matter searches
  -
  -

What's next?

- Precision:
  - Beyond the dipole approximation
  - Hyperfine transitions
  - Long range forces?

- Applications:
  - Dark matter searches
  - Astrophysics
  -

What's next?

- Precision:
  - Beyond the dipole approximation
  - Hyperfine transitions
  - Long range forces?

- Applications:
  - Dark matter searches
  - Astrophysics
  - Black hole superradiance?

Summary

- Evidence for dark matter at all scales, little idea as to its fundamental nature

Summary

- Evidence for dark matter at all scales, little idea as to its fundamental nature

- Hydrogen an incredibly precise tool

Summary

- Evidence for dark matter at all scales, little idea as to its fundamental nature
- Hydrogen an incredibly precise tool
- Generalised hydrogen transitions could search for DM

Summary

- Evidence for dark matter at all scales, little idea as to its fundamental nature
- Hydrogen an incredibly precise tool
- Generalised hydrogen transitions could search for DM
- CINCO provides a quick and precise way of computing amplitudes

Thank you!
Questions?